MA 229/MA 235 - Lecture 22

IISc

Recap

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- Tensor fields and Riemannian metrics.

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- Elementary alternating tensors.

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- More generally, Theorem: For any two multi-indices I, J, $\epsilon^I \wedge \epsilon^J = \epsilon^{IJ}$.

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- Using this property, we can calculate pullbacks for several examples.

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- In particular, $d\tilde{x}^1 \wedge \dots = \det(\frac{\partial \tilde{x}^i}{\partial x^j}) dx^1 \wedge \dots$.