MA 229/MA 235 - Lecture 7

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Recap

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- No need to worry about 'maximal'.
- There is a countable basis of smooth coordinate balls.
- Examples of smooth manifolds (including spheres).

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Smooth manifolds-with-boundary

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- A smooth atlas

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- A smooth atlas on a topological manifold-with-boundary

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- They are compatible with the usual charts too (exercise) and hence define the same smooth structure.
- In a sense, they are related to the notion that S^n is the one-point compactification of $\mathbb{R}^n$.