MA 229/MA 235 - Lecture 23

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Recap

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- Wedge product and its properties.

Differential forms on manifolds

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- Recall that $F^*df = dF^*f$. Moreover, if $\omega = \omega_i dx^i$, then $F^*\omega = \omega_i \circ F dF^i$.
- For k-forms, the pullback is $\mathbb{R}$-linear (why?).

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- In particular, $d\tilde{x}^1 \wedge \dots = \det(\frac{\partial \tilde{x}^i}{\partial x^j}) dx^1 \wedge \dots$.

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- It coincides with the usual curl in $\mathbb{R}^3$.

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- De Rham cohomology: $H^k(M) = \frac{closed\ \ k-forms}{exact\ \ ones}$.
- $H^0(M) = \mathbb{R}^k$ where k is the number of connected components (why?)
- $H^1(\mathbb{R}^2 - 0) = \mathbb{R}$. $H^k(M) = 0$ when $k > n$ (why?) $H^k(\mathbb{R}^n) = 0$ for $k > 0$.
- It turns out that the de Rham cohomology coincides with singular cohomology. So it is invariant under homeomorphism. (Thus showing how hard it is to distinguish between smooth structures.)