MA 229/MA 235 - Lecture 25

IISc

Recap

Recap

- Change of variables formula.

Recap

- Change of variables formula.
- Integration of top forms in $\mathbb{R}^n$

Integration of functions vs forms

Integration of functions vs forms

- Recall that it makes

Integration of functions vs forms

- Recall that it makes no sense to try to define

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold because when we change coordinates,

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold because when we change coordinates, the integral does not remain invariant.

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold because when we change coordinates, the integral does not remain invariant.
- On the other hand,

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold because when we change coordinates, the integral does not remain invariant.
- On the other hand, in $\mathbb{R}^n$ we can define the integrals of top forms.

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold because when we change coordinates, the integral does not remain invariant.
- On the other hand, in $\mathbb{R}^n$ we can define the integrals of top forms. So we could try

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold because when we change coordinates, the integral does not remain invariant.
- On the other hand, in $\mathbb{R}^n$ we can define the integrals of top forms. So we could try $\int_M \omega = \sum_i \int_{\mathbb{R}^n} \rho_i f dx^1 dx^2 \dots$.

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold because when we change coordinates, the integral does not remain invariant.
- On the other hand, in $\mathbb{R}^n$ we can define the integrals of top forms. So we could try $\int_M \omega = \sum_i \int_{\mathbb{R}^n} \rho_i f dx^1 dx^2 \ldots$. The only problem is that

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold because when we change coordinates, the integral does not remain invariant.
- On the other hand, in $\mathbb{R}^n$ we can define the integrals of top forms. So we could try $\int_M \omega = \sum_i \int_{\mathbb{R}^n} \rho_i f dx^1 dx^2 \ldots$. The only problem is that the sign of the Jacobian plays a role in the change of variables formula.

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold because when we change coordinates, the integral does not remain invariant.
- On the other hand, in $\mathbb{R}^n$ we can define the integrals of top forms. So we could try $\int_M \omega = \sum_i \int_{\mathbb{R}^n} \rho_i f dx^1 dx^2 \ldots$. The only problem is that the sign of the Jacobian plays a role in the change of variables formula.
- What if we could cover

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold because when we change coordinates, the integral does not remain invariant.
- On the other hand, in $\mathbb{R}^n$ we can define the integrals of top forms. So we could try $\int_M \omega = \sum_i \int_{\mathbb{R}^n} \rho_i f dx^1 dx^2 \ldots$. The only problem is that the sign of the Jacobian plays a role in the change of variables formula.
- What if we could cover M by coordinate charts

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold because when we change coordinates, the integral does not remain invariant.
- On the other hand, in $\mathbb{R}^n$ we can define the integrals of top forms. So we could try $\int_M \omega = \sum_i \int_{\mathbb{R}^n} \rho_i f dx^1 dx^2 \ldots$. The only problem is that the sign of the Jacobian plays a role in the change of variables formula.
- What if we could cover M by coordinate charts such that the Jacobians are all positive?

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold because when we change coordinates, the integral does not remain invariant.
- On the other hand, in $\mathbb{R}^n$ we can define the integrals of top forms. So we could try $\int_M \omega = \sum_i \int_{\mathbb{R}^n} \rho_i f dx^1 dx^2 \ldots$. The only problem is that the sign of the Jacobian plays a role in the change of variables formula.
- What if we could cover M by coordinate charts such that the Jacobians are all positive? In this case, we have some hope.

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold because when we change coordinates, the integral does not remain invariant.
- On the other hand, in $\mathbb{R}^n$ we can define the integrals of top forms. So we could try $\int_M \omega = \sum_i \int_{\mathbb{R}^n} \rho_i f dx^1 dx^2 \ldots$. The only problem is that the sign of the Jacobian plays a role in the change of variables formula.
- What if we could cover M by coordinate charts such that the Jacobians are all positive? In this case, we have some hope.
- "Def" (Warning:

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold because when we change coordinates, the integral does not remain invariant.
- On the other hand, in $\mathbb{R}^n$ we can define the integrals of top forms. So we could try $\int_M \omega = \sum_i \int_{\mathbb{R}^n} \rho_i f dx^1 dx^2 \ldots$. The only problem is that the sign of the Jacobian plays a role in the change of variables formula.
- What if we could cover M by coordinate charts such that the Jacobians are all positive? In this case, we have some hope.
- "Def" (Warning: This definition is useful when $dim(M) > 1$ or $\partial M = \phi$.):

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold because when we change coordinates, the integral does not remain invariant.
- On the other hand, in $\mathbb{R}^n$ we can define the integrals of top forms. So we could try $\int_M \omega = \sum_i \int_{\mathbb{R}^n} \rho_i f dx^1 dx^2 \ldots$. The only problem is that the sign of the Jacobian plays a role in the change of variables formula.
- What if we could cover M by coordinate charts such that the Jacobians are all positive? In this case, we have some hope.
- "Def" (Warning: This definition is useful when $dim(M) > 1$ or $\partial M = \phi$.): Suppose M is a smooth manifold (with or without boundary)

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold because when we change coordinates, the integral does not remain invariant.
- On the other hand, in $\mathbb{R}^n$ we can define the integrals of top forms. So we could try $\int_M \omega = \sum_i \int_{\mathbb{R}^n} \rho_i f dx^1 dx^2 \ldots$. The only problem is that the sign of the Jacobian plays a role in the change of variables formula.
- What if we could cover M by coordinate charts such that the Jacobians are all positive? In this case, we have some hope.
- "Def" (Warning: This definition is useful when $dim(M) > 1$ or $\partial M = \phi$.): Suppose M is a smooth manifold (with or without boundary) and (x_α, U_α) is a smooth atlas consisting of connected charts

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold because when we change coordinates, the integral does not remain invariant.
- On the other hand, in $\mathbb{R}^n$ we can define the integrals of top forms. So we could try $\int_M \omega = \sum_i \int_{\mathbb{R}^n} \rho_i f dx^1 dx^2 \ldots$. The only problem is that the sign of the Jacobian plays a role in the change of variables formula.
- What if we could cover M by coordinate charts such that the Jacobians are all positive? In this case, we have some hope.
- "Def" (Warning: This definition is useful when $dim(M) > 1$ or $\partial M = \phi$.): Suppose M is a smooth manifold (with or without boundary) and (x_α, U_α) is a smooth atlas consisting of connected charts such that $\det(\frac{\partial x^i_\alpha}{\partial x^j_\beta}) > 0$ *on* $U_\alpha \cap U_\beta$ for all α, β,

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold because when we change coordinates, the integral does not remain invariant.
- On the other hand, in $\mathbb{R}^n$ we can define the integrals of top forms. So we could try $\int_M \omega = \sum_i \int_{\mathbb{R}^n} \rho_i f dx^1 dx^2 \ldots$. The only problem is that the sign of the Jacobian plays a role in the change of variables formula.
- What if we could cover M by coordinate charts such that the Jacobians are all positive? In this case, we have some hope.
- "Def" (Warning: This definition is useful when $dim(M) > 1$ or $\partial M = \phi$.): Suppose M is a smooth manifold (with or without boundary) and (x_α, U_α) is a smooth atlas consisting of connected charts such that $\det(\frac{\partial x^i_\alpha}{\partial x^j_\beta}) > 0 \; on \; U_\alpha \cap U_\beta$ for all α, β, then we say that M is equipped with an oriented atlas/

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold because when we change coordinates, the integral does not remain invariant.
- On the other hand, in $\mathbb{R}^n$ we can define the integrals of top forms. So we could try $\int_M \omega = \sum_i \int_{\mathbb{R}^n} \rho_i f dx^1 dx^2 \ldots$. The only problem is that the sign of the Jacobian plays a role in the change of variables formula.
- What if we could cover M by coordinate charts such that the Jacobians are all positive? In this case, we have some hope.
- "Def" (Warning: This definition is useful when $dim(M) > 1$ or $\partial M = \phi$.): Suppose M is a smooth manifold (with or without boundary) and (x_α, U_α) is a smooth atlas consisting of connected charts such that $\det(\frac{\partial x_\alpha^i}{\partial x_\beta^j}) > 0$ *on* $U_\alpha \cap U_\beta$ for all α, β, then we say that M is equipped with an oriented atlas/ M has a given orientation. (

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold because when we change coordinates, the integral does not remain invariant.
- On the other hand, in $\mathbb{R}^n$ we can define the integrals of top forms. So we could try $\int_M \omega = \sum_i \int_{\mathbb{R}^n} \rho_i f dx^1 dx^2 \ldots$. The only problem is that the sign of the Jacobian plays a role in the change of variables formula.
- What if we could cover M by coordinate charts such that the Jacobians are all positive? In this case, we have some hope.
- "Def" (Warning: This definition is useful when $dim(M) > 1$ or $\partial M = \phi$.): Suppose M is a smooth manifold (with or without boundary) and (x_α, U_α) is a smooth atlas consisting of connected charts such that $\det(\frac{\partial x^i_\alpha}{\partial x^j_\beta}) > 0$ *on* $U_\alpha \cap U_\beta$ for all α, β, then we say that M is equipped with an oriented atlas/ M has a given orientation. (If such an atlas exists,

Integration of functions vs forms

- Recall that it makes no sense to try to define the integral of a function $f : M \to \mathbb{R}$ on a manifold because when we change coordinates, the integral does not remain invariant.
- On the other hand, in $\mathbb{R}^n$ we can define the integrals of top forms. So we could try $\int_M \omega = \sum_i \int_{\mathbb{R}^n} \rho_i f dx^1 dx^2 \ldots$. The only problem is that the sign of the Jacobian plays a role in the change of variables formula.
- What if we could cover M by coordinate charts such that the Jacobians are all positive? In this case, we have some hope.
- "Def" (Warning: This definition is useful when $dim(M) > 1$ or $\partial M = \phi$.): Suppose M is a smooth manifold (with or without boundary) and (x_α, U_α) is a smooth atlas consisting of connected charts such that $\det(\frac{\partial x_\alpha^i}{\partial x_\beta^j}) > 0$ *on* $U_\alpha \cap U_\beta$ for all α, β, then we say that M is equipped with an oriented atlas/ M has a given orientation. (If such an atlas exists, then we say that M is orientable.)

Orientation

Orientation

- When do we say that

Orientation

- When do we say that two such atlases give the

Orientation

- When do we say that two such atlases give the "same" orientation?

Orientation

- When do we say that two such atlases give the "same" orientation?
- Def:

Orientation

- When do we say that two such atlases give the "same" orientation?
- Def: Two smooth oriented atlases $\mathcal{A}$ and $\mathcal{B}$ are

Orientation

- When do we say that two such atlases give the "same" orientation?
- Def: Two smooth oriented atlases $\mathcal{A}$ and $\mathcal{B}$ are said to be compatible orientation-wise/

Orientation

- When do we say that two such atlases give the "same" orientation?
- Def: Two smooth oriented atlases $\mathcal{A}$ and $\mathcal{B}$ are said to be compatible orientation-wise/define the same orientation if

Orientation

- When do we say that two such atlases give the "same" orientation?
- Def: Two smooth oriented atlases $\mathcal{A}$ and $\mathcal{B}$ are said to be compatible orientation-wise/define the same orientation if $\mathcal{A} \cup \mathcal{B}$ is an oriented atlas.

Orientation

- When do we say that two such atlases give the "same" orientation?
- Def: Two smooth oriented atlases $\mathcal{A}$ and $\mathcal{B}$ are said to be compatible orientation-wise/define the same orientation if $\mathcal{A} \cup \mathcal{B}$ is an oriented atlas.
- Suppose M is orientable.

Orientation

- When do we say that two such atlases give the "same" orientation?
- Def: Two smooth oriented atlases $\mathcal{A}$ and $\mathcal{B}$ are said to be compatible orientation-wise/define the same orientation if $\mathcal{A} \cup \mathcal{B}$ is an oriented atlas.
- Suppose M is orientable. Then orientation-compatibility is an equivalence relation among oriented atlases (why?)

Orientation

- When do we say that two such atlases give the "same" orientation?
- Def: Two smooth oriented atlases $\mathcal{A}$ and $\mathcal{B}$ are said to be compatible orientation-wise/define the same orientation if $\mathcal{A} \cup \mathcal{B}$ is an oriented atlas.
- Suppose M is orientable. Then orientation-compatibility is an equivalence relation among oriented atlases (why?)
- To determine

Orientation

- When do we say that two such atlases give the "same" orientation?
- Def: Two smooth oriented atlases $\mathcal{A}$ and $\mathcal{B}$ are said to be compatible orientation-wise/define the same orientation if $\mathcal{A} \cup \mathcal{B}$ is an oriented atlas.
- Suppose M is orientable. Then orientation-compatibility is an equivalence relation among oriented atlases (why?)
- To determine the number of equivalence classes,

Orientation

- When do we say that two such atlases give the "same" orientation?
- Def: Two smooth oriented atlases $\mathcal{A}$ and $\mathcal{B}$ are said to be compatible orientation-wise/define the same orientation if $\mathcal{A} \cup \mathcal{B}$ is an oriented atlas.
- Suppose M is orientable. Then orientation-compatibility is an equivalence relation among oriented atlases (why?)
- To determine the number of equivalence classes, we need a more concise interpretation of orientation.

Orientation through top forms

Orientation through top forms

- Given an oriented manifold $(M, (x_\alpha, U_\alpha))$,

Orientation through top forms

- Given an oriented manifold $(M, (x_\alpha, U_\alpha))$, let ρ_α be a partition-of-unity subordinate to the atlas.

Orientation through top forms

- Given an oriented manifold $(M, (x_\alpha, U_\alpha))$, let ρ_α be a partition-of-unity subordinate to the atlas. Define $\omega = \sum_\alpha \rho_\alpha dx_\alpha^1 \wedge dx_\alpha^2 \ldots$.

Orientation through top forms

- Given an oriented manifold $(M, (x_\alpha, U_\alpha))$, let ρ_α be a partition-of-unity subordinate to the atlas. Define $\omega = \sum_\alpha \rho_\alpha dx^1_\alpha \wedge dx^2_\alpha \ldots$. Note that $\omega \neq 0$ anywhere (why?)

Orientation through top forms

- Given an oriented manifold $(M, (x_\alpha, U_\alpha))$, let ρ_α be a partition-of-unity subordinate to the atlas. Define $\omega = \sum_\alpha \rho_\alpha dx_\alpha^1 \wedge dx_\alpha^2 \ldots$. Note that $\omega \neq 0$ anywhere (why?) Moreover, ω is a positive multiple of

Orientation through top forms

- Given an oriented manifold $(M, (x_\alpha, U_\alpha))$, let ρ_α be a partition-of-unity subordinate to the atlas. Define $\omega = \sum_\alpha \rho_\alpha dx_\alpha^1 \wedge dx_\alpha^2 \ldots$. Note that $\omega \neq 0$ anywhere (why?) Moreover, ω is a positive multiple of $dx_\alpha^1 \wedge dx_\alpha^2 \ldots$ for all α.

Orientation through top forms

- Given an oriented manifold $(M, (x_\alpha, U_\alpha))$, let ρ_α be a partition-of-unity subordinate to the atlas. Define $\omega = \sum_\alpha \rho_\alpha dx_\alpha^1 \wedge dx_\alpha^2 \ldots$. Note that $\omega \neq 0$ anywhere (why?) Moreover, ω is a positive multiple of $dx_\alpha^1 \wedge dx_\alpha^2 \ldots$ for all α.
- Conversely, suppose M either does not have a boundary or $dim(M) > 1$.

Orientation through top forms

- Given an oriented manifold $(M, (x_\alpha, U_\alpha))$, let ρ_α be a partition-of-unity subordinate to the atlas. Define $\omega = \sum_\alpha \rho_\alpha dx_\alpha^1 \wedge dx_\alpha^2 \ldots$. Note that $\omega \neq 0$ anywhere (why?) Moreover, ω is a positive multiple of $dx_\alpha^1 \wedge dx_\alpha^2 \ldots$ for all α.
- Conversely, suppose M either does not have a boundary or $dim(M) > 1$. Also suppose ω is a nowhere vanishing top form,

Orientation through top forms

- Given an oriented manifold $(M, (x_\alpha, U_\alpha))$, let ρ_α be a partition-of-unity subordinate to the atlas. Define $\omega = \sum_\alpha \rho_\alpha dx_\alpha^1 \wedge dx_\alpha^2 \ldots$. Note that $\omega \neq 0$ anywhere (why?) Moreover, ω is a positive multiple of $dx_\alpha^1 \wedge dx_\alpha^2 \ldots$ for all α.
- Conversely, suppose M either does not have a boundary or $dim(M) > 1$. Also suppose ω is a nowhere vanishing top form, and suppose $\mathcal{A}$ is any atlas consisting of connected charts.

Orientation through top forms

- Given an oriented manifold $(M, (x_\alpha, U_\alpha))$, let ρ_α be a partition-of-unity subordinate to the atlas. Define $\omega = \sum_\alpha \rho_\alpha dx_\alpha^1 \wedge dx_\alpha^2 \ldots$. Note that $\omega \neq 0$ anywhere (why?) Moreover, ω is a positive multiple of $dx_\alpha^1 \wedge dx_\alpha^2 \ldots$ for all α.
- Conversely, suppose M either does not have a boundary or $dim(M) > 1$. Also suppose ω is a nowhere vanishing top form, and suppose $\mathcal{A}$ is any atlas consisting of connected charts. We can change the charts and produce a new atlas

Orientation through top forms

- Given an oriented manifold $(M, (x_\alpha, U_\alpha))$, let ρ_α be a partition-of-unity subordinate to the atlas. Define $\omega = \sum_\alpha \rho_\alpha dx_\alpha^1 \wedge dx_\alpha^2 \ldots$. Note that $\omega \neq 0$ anywhere (why?) Moreover, ω is a positive multiple of $dx_\alpha^1 \wedge dx_\alpha^2 \ldots$ for all α.
- Conversely, suppose M either does not have a boundary or $dim(M) > 1$. Also suppose ω is a nowhere vanishing top form, and suppose $\mathcal{A}$ is any atlas consisting of connected charts. We can change the charts and produce a new atlas to make sure that

Orientation through top forms

- Given an oriented manifold $(M, (x_\alpha, U_\alpha))$, let ρ_α be a partition-of-unity subordinate to the atlas. Define $\omega = \sum_\alpha \rho_\alpha dx_\alpha^1 \wedge dx_\alpha^2 \ldots$. Note that $\omega \neq 0$ anywhere (why?) Moreover, ω is a positive multiple of $dx_\alpha^1 \wedge dx_\alpha^2 \ldots$ for all α.
- Conversely, suppose M either does not have a boundary or $dim(M) > 1$. Also suppose ω is a nowhere vanishing top form, and suppose $\mathcal{A}$ is any atlas consisting of connected charts. We can change the charts and produce a new atlas to make sure that $\omega(\frac{\partial}{\partial x_\alpha^1}, \frac{\partial}{\partial x_\alpha^2}, \ldots) > 0$ for all α (how?)

Orientation through top forms

- Given an oriented manifold $(M,(x_\alpha, U_\alpha))$, let ρ_α be a partition-of-unity subordinate to the atlas. Define $\omega = \sum_\alpha \rho_\alpha dx_\alpha^1 \wedge dx_\alpha^2 \ldots$. Note that $\omega \neq 0$ anywhere (why?) Moreover, ω is a positive multiple of $dx_\alpha^1 \wedge dx_\alpha^2 \ldots$ for all α.
- Conversely, suppose M either does not have a boundary or $dim(M) > 1$. Also suppose ω is a nowhere vanishing top form, and suppose $\mathcal{A}$ is any atlas consisting of connected charts. We can change the charts and produce a new atlas to make sure that $\omega(\frac{\partial}{\partial x_\alpha^1}, \frac{\partial}{\partial x_\alpha^2}, \ldots) > 0$ for all α (how?)
- So a manifold (such that $dim(M) > 1$ or $\partial M = \phi$)

Orientation through top forms

- Given an oriented manifold $(M, (x_\alpha, U_\alpha))$, let ρ_α be a partition-of-unity subordinate to the atlas. Define $\omega = \sum_\alpha \rho_\alpha dx_\alpha^1 \wedge dx_\alpha^2 \ldots$. Note that $\omega \neq 0$ anywhere (why?) Moreover, ω is a positive multiple of $dx_\alpha^1 \wedge dx_\alpha^2 \ldots$ for all α.
- Conversely, suppose M either does not have a boundary or $dim(M) > 1$. Also suppose ω is a nowhere vanishing top form, and suppose $\mathcal{A}$ is any atlas consisting of connected charts. We can change the charts and produce a new atlas to make sure that $\omega(\frac{\partial}{\partial x_\alpha^1}, \frac{\partial}{\partial x_\alpha^2}, \ldots) > 0$ for all α (how?)
- So a manifold (such that $dim(M) > 1$ or $\partial M = \phi$) is orientable iff

Orientation through top forms

- Given an oriented manifold $(M,(x_\alpha,U_\alpha))$, let ρ_α be a partition-of-unity subordinate to the atlas. Define $\omega=\sum_\alpha \rho_\alpha dx_\alpha^1\wedge dx_\alpha^2\ldots$. Note that $\omega\neq 0$ anywhere (why?) Moreover, ω is a positive multiple of $dx_\alpha^1\wedge dx_\alpha^2\ldots$ for all α.
- Conversely, suppose M either does not have a boundary or $dim(M)>1$. Also suppose ω is a nowhere vanishing top form, and suppose $\mathcal{A}$ is any atlas consisting of connected charts. We can change the charts and produce a new atlas to make sure that $\omega(\frac{\partial}{\partial x_\alpha^1},\frac{\partial}{\partial x_\alpha^2},\ldots)>0$ for all α (how?)
- So a manifold (such that $dim(M)>1$ or $\partial M=\phi$) is orientable iff it admits a nowhere vanishing top form.

Orientation through top forms

- Given an oriented manifold $(M, (x_\alpha, U_\alpha))$, let ρ_α be a partition-of-unity subordinate to the atlas. Define $\omega = \sum_\alpha \rho_\alpha dx^1_\alpha \wedge dx^2_\alpha \ldots$. Note that $\omega \neq 0$ anywhere (why?) Moreover, ω is a positive multiple of $dx^1_\alpha \wedge dx^2_\alpha \ldots$ for all α.
- Conversely, suppose M either does not have a boundary or $dim(M) > 1$. Also suppose ω is a nowhere vanishing top form, and suppose $\mathcal{A}$ is any atlas consisting of connected charts. We can change the charts and produce a new atlas to make sure that $\omega(\frac{\partial}{\partial x^1_\alpha}, \frac{\partial}{\partial x^2_\alpha}, \ldots) > 0$ for all α (how?)
- So a manifold (such that $dim(M) > 1$ or $\partial M = \phi$) is orientable iff it admits a nowhere vanishing top form. We say that a chart is compatible with

Orientation through top forms

- Given an oriented manifold $(M, (x_\alpha, U_\alpha))$, let ρ_α be a partition-of-unity subordinate to the atlas. Define $\omega = \sum_\alpha \rho_\alpha dx_\alpha^1 \wedge dx_\alpha^2 \ldots$. Note that $\omega \neq 0$ anywhere (why?) Moreover, ω is a positive multiple of $dx_\alpha^1 \wedge dx_\alpha^2 \ldots$ for all α.
- Conversely, suppose M either does not have a boundary or $dim(M) > 1$. Also suppose ω is a nowhere vanishing top form, and suppose $\mathcal{A}$ is any atlas consisting of connected charts. We can change the charts and produce a new atlas to make sure that $\omega(\frac{\partial}{\partial x_\alpha^1}, \frac{\partial}{\partial x_\alpha^2}, \ldots) > 0$ for all α (how?)
- So a manifold (such that $dim(M) > 1$ or $\partial M = \phi$) is orientable iff it admits a nowhere vanishing top form. We say that a chart is compatible with an orientation form ω if

Orientation through top forms

- Given an oriented manifold $(M, (x_\alpha, U_\alpha))$, let ρ_α be a partition-of-unity subordinate to the atlas. Define $\omega = \sum_\alpha \rho_\alpha dx_\alpha^1 \wedge dx_\alpha^2 \ldots$. Note that $\omega \neq 0$ anywhere (why?) Moreover, ω is a positive multiple of $dx_\alpha^1 \wedge dx_\alpha^2 \ldots$ for all α.
- Conversely, suppose M either does not have a boundary or $dim(M) > 1$. Also suppose ω is a nowhere vanishing top form, and suppose $\mathcal{A}$ is any atlas consisting of connected charts. We can change the charts and produce a new atlas to make sure that $\omega(\frac{\partial}{\partial x_\alpha^1}, \frac{\partial}{\partial x_\alpha^2}, \ldots) > 0$ for all α (how?)
- So a manifold (such that $dim(M) > 1$ or $\partial M = \phi$) is orientable iff it admits a nowhere vanishing top form. We say that a chart is compatible with an orientation form ω if $\omega(\frac{\partial}{\partial x^1}, \ldots) > 0$ at all points.

Orientation through top forms

Orientation through top forms

- In fact,

Orientation through top forms

- In fact, define an equivalence relation between nowhere vanishing top forms:

Orientation through top forms

- In fact, define an equivalence relation between nowhere vanishing top forms: $\omega \sim \omega'$ if $\omega = f\omega'$ where $f > 0$.

Orientation through top forms

- In fact, define an equivalence relation between nowhere vanishing top forms: $\omega \sim \omega'$ if $\omega = f\omega'$ where $f > 0$. Then if M is connected we have exactly two equivalence classes (why?)

Orientation through top forms

- In fact, define an equivalence relation between nowhere vanishing top forms: $\omega \sim \omega'$ if $\omega = f\omega'$ where $f > 0$. Then if M is connected we have exactly two equivalence classes (why?)
- The above correspondence gives a bijection between

Orientation through top forms

- In fact, define an equivalence relation between nowhere vanishing top forms: $\omega \sim \omega'$ if $\omega = f\omega'$ where $f > 0$. Then if M is connected we have exactly two equivalence classes (why?)
- The above correspondence gives a bijection between the two sets of equivalence classes when M has no boundary or when $dim(M) > 1$, i.e.,

Orientation through top forms

- In fact, define an equivalence relation between nowhere vanishing top forms: $\omega \sim \omega'$ if $\omega = f\omega'$ where $f > 0$. Then if M is connected we have exactly two equivalence classes (why?)
- The above correspondence gives a bijection between the two sets of equivalence classes when M has no boundary or when $dim(M) > 1$, i.e., Given $[(x_\alpha, U_\alpha)]$ consider $[\sum_\alpha \rho_\alpha dx^1_\alpha \wedge \ldots]$.

Orientation through top forms

- In fact, define an equivalence relation between nowhere vanishing top forms: $\omega \sim \omega'$ if $\omega = f\omega'$ where $f > 0$. Then if M is connected we have exactly two equivalence classes (why?)
- The above correspondence gives a bijection between the two sets of equivalence classes when M has no boundary or when $dim(M) > 1$, i.e., Given $[(x_\alpha, U_\alpha)]$ consider $[\sum_\alpha \rho_\alpha dx_\alpha^1 \wedge \ldots]$. Firstly, this map is well-defined.

Orientation through top forms

- In fact, define an equivalence relation between nowhere vanishing top forms: $\omega \sim \omega'$ if $\omega = f\omega'$ where $f > 0$. Then if M is connected we have exactly two equivalence classes (why?)
- The above correspondence gives a bijection between the two sets of equivalence classes when M has no boundary or when $dim(M) > 1$, i.e., Given $[(x_\alpha, U_\alpha)]$ consider $[\sum_\alpha \rho_\alpha dx_\alpha^1 \wedge \ldots]$. Firstly, this map is well-defined. Secondly, it is onto (why?)

Orientation through top forms

- In fact, define an equivalence relation between nowhere vanishing top forms: $\omega \sim \omega'$ if $\omega = f\omega'$ where $f > 0$. Then if M is connected we have exactly two equivalence classes (why?)
- The above correspondence gives a bijection between the two sets of equivalence classes when M has no boundary or when $dim(M) > 1$, i.e., Given $[(x_\alpha, U_\alpha)]$ consider $[\sum_\alpha \rho_\alpha dx_\alpha^1 \wedge \ldots]$. Firstly, this map is well-defined. Secondly, it is onto (why?) Thirdly, it is $1-1$:

Orientation through top forms

- In fact, define an equivalence relation between nowhere vanishing top forms: $\omega \sim \omega'$ if $\omega = f\omega'$ where $f > 0$. Then if M is connected we have exactly two equivalence classes (why?)
- The above correspondence gives a bijection between the two sets of equivalence classes when M has no boundary or when $dim(M) > 1$, i.e., Given $[(x_\alpha, U_\alpha)]$ consider $[\sum_\alpha \rho_\alpha dx^1_\alpha \wedge \ldots]$. Firstly, this map is well-defined. Secondly, it is onto (why?) Thirdly, it is $1-1$: If $\frac{\sum_\alpha \rho_\alpha dx^1_\alpha \wedge \ldots}{\sum_{\alpha'} \rho'_\alpha dy^1_{\alpha'} \wedge \ldots} > 0$,

Orientation through top forms

- In fact, define an equivalence relation between nowhere vanishing top forms: $\omega \sim \omega'$ if $\omega = f\omega'$ where $f > 0$. Then if M is connected we have exactly two equivalence classes (why?)
- The above correspondence gives a bijection between the two sets of equivalence classes when M has no boundary or when $dim(M) > 1$, i.e., Given $[(x_\alpha, U_\alpha)]$ consider $[\sum_\alpha \rho_\alpha dx_\alpha^1 \wedge \ldots]$. Firstly, this map is well-defined. Secondly, it is onto (why?) Thirdly, it is $1-1$: If $\frac{\sum_\alpha \rho_\alpha dx_\alpha^1 \wedge \ldots}{\sum_{\alpha'} \rho'_\alpha dy_{\alpha'}^1 \wedge \ldots} > 0$, and if these two atlases are not compatible then

- In fact, define an equivalence relation between nowhere vanishing top forms: $\omega \sim \omega'$ if $\omega = f\omega'$ where $f > 0$. Then if M is connected we have exactly two equivalence classes (why?)
- The above correspondence gives a bijection between the two sets of equivalence classes when M has no boundary or when $dim(M) > 1$, i.e., Given $[(x_\alpha, U_\alpha)]$ consider $[\sum_\alpha \rho_\alpha dx_\alpha^1 \wedge \ldots]$. Firstly, this map is well-defined. Secondly, it is onto (why?) Thirdly, it is $1-1$: If $\frac{\sum_\alpha \rho_\alpha dx_\alpha^1 \wedge \ldots}{\sum_{\alpha'} \rho'_\alpha dy_{\alpha'}^1 \wedge \ldots} > 0$, and if these two atlases are not compatible then $\det(\frac{\partial x_\alpha^i}{\partial y_{\beta'}^j}) < 0$ for some α, β' throughout $U_\alpha \cap U_{\beta'}$ (why?).

Orientation through top forms

- In fact, define an equivalence relation between nowhere vanishing top forms: $\omega \sim \omega'$ if $\omega = f\omega'$ where $f > 0$. Then if M is connected we have exactly two equivalence classes (why?)
- The above correspondence gives a bijection between the two sets of equivalence classes when M has no boundary or when $dim(M) > 1$, i.e., Given $[(x_\alpha, U_\alpha)]$ consider $[\sum_\alpha \rho_\alpha dx_\alpha^1 \wedge \ldots]$. Firstly, this map is well-defined. Secondly, it is onto (why?) Thirdly, it is $1-1$: If $\frac{\sum_\alpha \rho_\alpha dx_\alpha^1 \wedge \ldots}{\sum_{\alpha'} \rho'_\alpha dy_{\alpha'}^1 \wedge \ldots} > 0$, and if these two atlases are not compatible then $\det(\frac{\partial x_\alpha^i}{\partial y_{\beta'}^j}) < 0$ for some α, β' throughout $U_\alpha \cap U_{\beta'}$ (why?). This means that the above ratio must be negative in this region (why?)

Orientation through top forms

- In fact, define an equivalence relation between nowhere vanishing top forms: $\omega \sim \omega'$ if $\omega = f\omega'$ where $f > 0$. Then if M is connected we have exactly two equivalence classes (why?)
- The above correspondence gives a bijection between the two sets of equivalence classes when M has no boundary or when $dim(M) > 1$, i.e., Given $[(x_\alpha, U_\alpha)]$ consider $[\sum_\alpha \rho_\alpha dx_\alpha^1 \wedge \ldots]$. Firstly, this map is well-defined. Secondly, it is onto (why?) Thirdly, it is $1-1$: If $\frac{\sum_\alpha \rho_\alpha dx_\alpha^1 \wedge \ldots}{\sum_{\alpha'} \rho'_\alpha dy_{\alpha'}^1 \wedge \ldots} > 0$, and if these two atlases are not compatible then $\det(\frac{\partial x_\alpha^i}{\partial y_{\beta'}^j}) < 0$ for some α, β' throughout $U_\alpha \cap U_{\beta'}$ (why?). This means that the above ratio must be negative in this region (why?) Thus we have a contradiction.

The case when $dim(M) = 1$ and $\partial M \neq \phi$

The case when $dim(M) = 1$ and $\partial M \neq \phi$

- The above correspondence means that

The case when $dim(M) = 1$ and $\partial M \neq \phi$

- The above correspondence means that we have exactly two equivalence classes for orientation when

The case when $dim(M) = 1$ and $\partial M \neq \phi$

- The above correspondence means that we have exactly two equivalence classes for orientation when $\partial M = \phi$ or $dim(M) > 1$.

The case when $dim(M) = 1$ and $\partial M \neq \phi$

- The above correspondence means that we have exactly two equivalence classes for orientation when $\partial M = \phi$ or $dim(M) > 1$. Often, one arbitrarily designates one class as "positively oriented"

The case when $dim(M) = 1$ and $\partial M \neq \phi$

- The above correspondence means that we have exactly two equivalence classes for orientation when $\partial M = \phi$ or $dim(M) > 1$. Often, one arbitrarily designates one class as "positively oriented" and the other as negatively oriented.

The case when $dim(M) = 1$ and $\partial M \neq \phi$

- The above correspondence means that we have exactly two equivalence classes for orientation when $\partial M = \phi$ or $dim(M) > 1$. Often, one arbitrarily designates one class as "positively oriented" and the other as negatively oriented.
- Unfortunately, in this case

The case when $dim(M) = 1$ and $\partial M \neq \phi$

- The above correspondence means that we have exactly two equivalence classes for orientation when $\partial M = \phi$ or $dim(M) > 1$. Often, one arbitrarily designates one class as "positively oriented" and the other as negatively oriented.
- Unfortunately, in this case since we have defined the boundary chart to have

The case when $dim(M) = 1$ and $\partial M \neq \phi$

- The above correspondence means that we have exactly two equivalence classes for orientation when $\partial M = \phi$ or $dim(M) > 1$. Often, one arbitrarily designates one class as "positively oriented" and the other as negatively oriented.
- Unfortunately, in this case since we have defined the boundary chart to have positive last coordinate,

The case when $dim(M) = 1$ and $\partial M \neq \phi$

- The above correspondence means that we have exactly two equivalence classes for orientation when $\partial M = \phi$ or $dim(M) > 1$. Often, one arbitrarily designates one class as "positively oriented" and the other as negatively oriented.
- Unfortunately, in this case since we have defined the boundary chart to have positive last coordinate, our definition of orientation is not a nice one.

The case when $dim(M) = 1$ and $\partial M \neq \phi$

- The above correspondence means that we have exactly two equivalence classes for orientation when $\partial M = \phi$ or $dim(M) > 1$. Often, one arbitrarily designates one class as "positively oriented" and the other as negatively oriented.
- Unfortunately, in this case since we have defined the boundary chart to have positive last coordinate, our definition of orientation is not a nice one. To avoid this problem,

The case when $dim(M) = 1$ and $\partial M \neq \phi$

- The above correspondence means that we have exactly two equivalence classes for orientation when $\partial M = \phi$ or $dim(M) > 1$. Often, one arbitrarily designates one class as "positively oriented" and the other as negatively oriented.
- Unfortunately, in this case since we have defined the boundary chart to have positive last coordinate, our definition of orientation is not a nice one. To avoid this problem, one *defines* orientation of manifolds using the existence of nowhere vanishing top forms.

The case when $dim(M) = 1$ and $\partial M \neq \phi$

- The above correspondence means that we have exactly two equivalence classes for orientation when $\partial M = \phi$ or $dim(M) > 1$. Often, one arbitrarily designates one class as "positively oriented" and the other as negatively oriented.
- Unfortunately, in this case since we have defined the boundary chart to have positive last coordinate, our definition of orientation is not a nice one. To avoid this problem, one *defines* orientation of manifolds using the existence of nowhere vanishing top forms. Then every orientable manifold (with or without boundary) has exactly two orientation classes.

The case when $dim(M) = 1$ and $\partial M \neq \phi$

- The above correspondence means that we have exactly two equivalence classes for orientation when $\partial M = \phi$ or $dim(M) > 1$. Often, one arbitrarily designates one class as "positively oriented" and the other as negatively oriented.
- Unfortunately, in this case since we have defined the boundary chart to have positive last coordinate, our definition of orientation is not a nice one. To avoid this problem, one *defines* orientation of manifolds using the existence of nowhere vanishing top forms. Then every orientable manifold (with or without boundary) has exactly two orientation classes. When $\partial M = \phi$ or $dim(M) > 1$,

The case when $dim(M) = 1$ and $\partial M \neq \phi$

- The above correspondence means that we have exactly two equivalence classes for orientation when $\partial M = \phi$ or $dim(M) > 1$. Often, one arbitrarily designates one class as "positively oriented" and the other as negatively oriented.
- Unfortunately, in this case since we have defined the boundary chart to have positive last coordinate, our definition of orientation is not a nice one. To avoid this problem, one *defines* orientation of manifolds using the existence of nowhere vanishing top forms. Then every orientable manifold (with or without boundary) has exactly two orientation classes. When $\partial M = \phi$ or $dim(M) > 1$, this corresponds to orienting using coordinate charts.

Examples of orientable manifolds

Examples of orientable manifolds

- $\mathbb{R}^n$ is orientable.

Examples of orientable manifolds

- $\mathbb{R}^n$ is orientable.
- A codimension-0 submanifold $D \subset M$ is orientable if M is so:

Examples of orientable manifolds

- $\mathbb{R}^n$ is orientable.
- A codimension-0 submanifold $D \subset M$ is orientable if M is so: Suppose ω is an orientation form on M,

Examples of orientable manifolds

- $\mathbb{R}^n$ is orientable.
- A codimension-0 submanifold $D \subset M$ is orientable if M is so: Suppose ω is an orientation form on M, then $i^*\omega$ is one on D.

Examples of orientable manifolds

- $\mathbb{R}^n$ is orientable.
- A codimension-0 submanifold $D \subset M$ is orientable if M is so: Suppose ω is an orientation form on M, then $i^*\omega$ is one on D.
- If M, N are orientable,

Examples of orientable manifolds

- $\mathbb{R}^n$ is orientable.
- A codimension-0 submanifold $D \subset M$ is orientable if M is so: Suppose ω is an orientation form on M, then $i^*\omega$ is one on D.
- If M, N are orientable, then so is $M \times N$ with the

Examples of orientable manifolds

- $\mathbb{R}^n$ is orientable.
- A codimension-0 submanifold $D \subset M$ is orientable if M is so: Suppose ω is an orientation form on M, then $i^*\omega$ is one on D.
- If M, N are orientable, then so is $M \times N$ with the "product orientation":

Examples of orientable manifolds

- $\mathbb{R}^n$ is orientable.
- A codimension-0 submanifold $D \subset M$ is orientable if M is so: Suppose ω is an orientation form on M, then $i^*\omega$ is one on D.
- If M, N are orientable, then so is $M \times N$ with the "product orientation": Take $\pi_1^*\omega_1 \wedge \pi_2^*\omega_2$ as the orientation form.

Examples of orientable manifolds

- $\mathbb{R}^n$ is orientable.
- A codimension-0 submanifold $D \subset M$ is orientable if M is so: Suppose ω is an orientation form on M, then $i^*\omega$ is one on D.
- If M, N are orientable, then so is $M \times N$ with the "product orientation": Take $\pi_1^*\omega_1 \wedge \pi_2^*\omega_2$ as the orientation form.
- Suppose $F : M \to N$ (where M, N are connected with $dim > 0$) is a smooth map

Examples of orientable manifolds

- $\mathbb{R}^n$ is orientable.
- A codimension-0 submanifold $D \subset M$ is orientable if M is so: Suppose ω is an orientation form on M, then $i^*\omega$ is one on D.
- If M, N are orientable, then so is $M \times N$ with the "product orientation": Take $\pi_1^*\omega_1 \wedge \pi_2^*\omega_2$ as the orientation form.
- Suppose $F : M \to N$ (where M, N are connected with $dim > 0$) is a smooth map such that F_* is invertible at all points.

Examples of orientable manifolds

- $\mathbb{R}^n$ is orientable.
- A codimension-0 submanifold $D \subset M$ is orientable if M is so: Suppose ω is an orientation form on M, then $i^*\omega$ is one on D.
- If M, N are orientable, then so is $M \times N$ with the "product orientation": Take $\pi_1^*\omega_1 \wedge \pi_2^*\omega_2$ as the orientation form.
- Suppose $F : M \to N$ (where M, N are connected with $dim > 0$) is a smooth map such that F_* is invertible at all points. If $(F_*)_p$ is orientation-preserving at all points,

Examples of orientable manifolds

- $\mathbb{R}^n$ is orientable.
- A codimension-0 submanifold $D \subset M$ is orientable if M is so: Suppose ω is an orientation form on M, then $i^*\omega$ is one on D.
- If M, N are orientable, then so is $M \times N$ with the "product orientation": Take $\pi_1^*\omega_1 \wedge \pi_2^*\omega_2$ as the orientation form.
- Suppose $F : M \to N$ (where M, N are connected with $dim > 0$) is a smooth map such that F_* is invertible at all points. If $(F_*)_p$ is orientation-preserving at all points, then F is said to be orientation-preserving.

Examples of orientable manifolds

- $\mathbb{R}^n$ is orientable.
- A codimension-0 submanifold $D \subset M$ is orientable if M is so: Suppose ω is an orientation form on M, then $i^*\omega$ is one on D.
- If M, N are orientable, then so is $M \times N$ with the "product orientation": Take $\pi_1^*\omega_1 \wedge \pi_2^*\omega_2$ as the orientation form.
- Suppose $F : M \to N$ (where M, N are connected with $dim > 0$) is a smooth map such that F_* is invertible at all points. If $(F_*)_p$ is orientation-preserving at all points, then F is said to be orientation-preserving. Otherwise it is said to be orientation-reversing.

Examples of orientable manifolds

- $\mathbb{R}^n$ is orientable.
- A codimension-0 submanifold $D \subset M$ is orientable if M is so: Suppose ω is an orientation form on M, then $i^*\omega$ is one on D.
- If M, N are orientable, then so is $M \times N$ with the "product orientation": Take $\pi_1^*\omega_1 \wedge \pi_2^*\omega_2$ as the orientation form.
- Suppose $F : M \to N$ (where M, N are connected with $dim > 0$) is a smooth map such that F_* is invertible at all points. If $(F_*)_p$ is orientation-preserving at all points, then F is said to be orientation-preserving. Otherwise it is said to be orientation-reversing. Given an orientation $[\omega]$ on N,

Examples of orientable manifolds

- $\mathbb{R}^n$ is orientable.
- A codimension-0 submanifold $D \subset M$ is orientable if M is so: Suppose ω is an orientation form on M, then $i^*\omega$ is one on D.
- If M, N are orientable, then so is $M \times N$ with the "product orientation": Take $\pi_1^*\omega_1 \wedge \pi_2^*\omega_2$ as the orientation form.
- Suppose $F : M \to N$ (where M, N are connected with $dim > 0$) is a smooth map such that F_* is invertible at all points. If $(F_*)_p$ is orientation-preserving at all points, then F is said to be orientation-preserving. Otherwise it is said to be orientation-reversing. Given an orientation $[\omega]$ on N, there is a unique orientation (

Examples of orientable manifolds

- $\mathbb{R}^n$ is orientable.
- A codimension-0 submanifold $D \subset M$ is orientable if M is so: Suppose ω is an orientation form on M, then $i^*\omega$ is one on D.
- If M, N are orientable, then so is $M \times N$ with the "product orientation": Take $\pi_1^*\omega_1 \wedge \pi_2^*\omega_2$ as the orientation form.
- Suppose $F : M \to N$ (where M, N are connected with $dim > 0$) is a smooth map such that F_* is invertible at all points. If $(F_*)_p$ is orientation-preserving at all points, then F is said to be orientation-preserving. Otherwise it is said to be orientation-reversing. Given an orientation $[\omega]$ on N, there is a unique orientation (called the pullback orientation)

Examples of orientable manifolds

- $\mathbb{R}^n$ is orientable.
- A codimension-0 submanifold $D \subset M$ is orientable if M is so: Suppose ω is an orientation form on M, then $i^*\omega$ is one on D.
- If M, N are orientable, then so is $M \times N$ with the "product orientation": Take $\pi_1^*\omega_1 \wedge \pi_2^*\omega_2$ as the orientation form.
- Suppose $F : M \to N$ (where M, N are connected with $dim > 0$) is a smooth map such that F_* is invertible at all points. If $(F_*)_p$ is orientation-preserving at all points, then F is said to be orientation-preserving. Otherwise it is said to be orientation-reversing. Given an orientation $[\omega]$ on N, there is a unique orientation (called the pullback orientation) such that F is orientation-preserving:

Examples of orientable manifolds

- $\mathbb{R}^n$ is orientable.
- A codimension-0 submanifold $D \subset M$ is orientable if M is so: Suppose ω is an orientation form on M, then $i^*\omega$ is one on D.
- If M, N are orientable, then so is $M \times N$ with the "product orientation": Take $\pi_1^*\omega_1 \wedge \pi_2^*\omega_2$ as the orientation form.
- Suppose $F : M \to N$ (where M, N are connected with $dim > 0$) is a smooth map such that F_* is invertible at all points. If $(F_*)_p$ is orientation-preserving at all points, then F is said to be orientation-preserving. Otherwise it is said to be orientation-reversing. Given an orientation $[\omega]$ on N, there is a unique orientation (called the pullback orientation) such that F is orientation-preserving: $[F^*\omega]$ does the job (why?).

Examples of orientable manifolds

- $\mathbb{R}^n$ is orientable.
- A codimension-0 submanifold $D \subset M$ is orientable if M is so: Suppose ω is an orientation form on M, then $i^*\omega$ is one on D.
- If M, N are orientable, then so is $M \times N$ with the "product orientation": Take $\pi_1^*\omega_1 \wedge \pi_2^*\omega_2$ as the orientation form.
- Suppose $F : M \to N$ (where M, N are connected with $dim > 0$) is a smooth map such that F_* is invertible at all points. If $(F_*)_p$ is orientation-preserving at all points, then F is said to be orientation-preserving. Otherwise it is said to be orientation-reversing. Given an orientation $[\omega]$ on N, there is a unique orientation (called the pullback orientation) such that F is orientation-preserving: $[F^*\omega]$ does the job (why?). If $[\eta]$ is any other such orientation,

Examples of orientable manifolds

- $\mathbb{R}^n$ is orientable.
- A codimension-0 submanifold $D \subset M$ is orientable if M is so: Suppose ω is an orientation form on M, then $i^*\omega$ is one on D.
- If M, N are orientable, then so is $M \times N$ with the "product orientation": Take $\pi_1^*\omega_1 \wedge \pi_2^*\omega_2$ as the orientation form.
- Suppose $F : M \to N$ (where M, N are connected with $dim > 0$) is a smooth map such that F_* is invertible at all points. If $(F_*)_p$ is orientation-preserving at all points, then F is said to be orientation-preserving. Otherwise it is said to be orientation-reversing. Given an orientation $[\omega]$ on N, there is a unique orientation (called the pullback orientation) such that F is orientation-preserving: $[F^*\omega]$ does the job (why?). If $[\eta]$ is any other such orientation, then $\omega(F_*e_1, F_*e_2, \ldots)/\eta(e_1, \ldots) > 0$ (why?). Thus $[\eta] = [F^*\omega]$.

Examples of orientable manifolds

Examples of orientable manifolds

- Hypersurfaces in M:

Examples of orientable manifolds

- Hypersurfaces in M: Suppose $(M, [\omega])$ is an oriented smooth manifold with or without boundary,

Examples of orientable manifolds

- Hypersurfaces in M: Suppose $(M, [\omega])$ is an oriented smooth manifold with or without boundary, and $S \subset M$ is a smooth hypersurface (without boundary that does not intersect ∂M).

- Hypersurfaces in M: Suppose $(M, [\omega])$ is an oriented smooth manifold with or without boundary, and $S \subset M$ is a smooth hypersurface (without boundary that does not intersect ∂M). Suppose $\vec{N}$ is a section of

Examples of orientable manifolds

- Hypersurfaces in M: Suppose $(M, [\omega])$ is an oriented smooth manifold with or without boundary, and $S \subset M$ is a smooth hypersurface (without boundary that does not intersect ∂M). Suppose $\vec{N}$ is a section of TM restricted to S such that

Examples of orientable manifolds

- Hypersurfaces in M: Suppose $(M, [\omega])$ is an oriented smooth manifold with or without boundary, and $S \subset M$ is a smooth hypersurface (without boundary that does not intersect ∂M). Suppose $\vec{N}$ is a section of TM restricted to S such that $\vec{N}$ is nowhere tangent to S.

Examples of orientable manifolds

- Hypersurfaces in M: Suppose $(M,[\omega])$ is an oriented smooth manifold with or without boundary, and $S \subset M$ is a smooth hypersurface (without boundary that does not intersect ∂M). Suppose $\vec{N}$ is a section of TM restricted to S such that $\vec{N}$ is nowhere tangent to S. Then S is orientable with

Examples of orientable manifolds

- Hypersurfaces in M: Suppose $(M, [\omega])$ is an oriented smooth manifold with or without boundary, and $S \subset M$ is a smooth hypersurface (without boundary that does not intersect ∂M). Suppose $\vec{N}$ is a section of TM restricted to S such that $\vec{N}$ is nowhere tangent to S. Then S is orientable with the orientation given by the form $(e_1, \ldots, e_{n-1}) \to \omega(\vec{N}, e_1, \ldots)$ (

Examples of orientable manifolds

- Hypersurfaces in M: Suppose $(M, [\omega])$ is an oriented smooth manifold with or without boundary, and $S \subset M$ is a smooth hypersurface (without boundary that does not intersect ∂M). Suppose $\vec{N}$ is a section of TM restricted to S such that $\vec{N}$ is nowhere tangent to S. Then S is orientable with the orientation given by the form $(e_1, \ldots, e_{n-1}) \to \omega(\vec{N}, e_1, \ldots)$ (Indeed, $\vec{N}, e_1 \ldots$ are linearly independent and hence

Examples of orientable manifolds

- Hypersurfaces in M: Suppose $(M, [\omega])$ is an oriented smooth manifold with or without boundary, and $S \subset M$ is a smooth hypersurface (without boundary that does not intersect ∂M). Suppose $\vec{N}$ is a section of TM restricted to S such that $\vec{N}$ is nowhere tangent to S. Then S is orientable with the orientation given by the form $(e_1, \ldots, e_{n-1}) \to \omega(\vec{N}, e_1, \ldots)$ (Indeed, $\vec{N}, e_1 \ldots$ are linearly independent and hence $\omega(\vec{N}, \ldots) \neq 0$.)

Examples of orientable manifolds

- Hypersurfaces in M: Suppose $(M, [\omega])$ is an oriented smooth manifold with or without boundary, and $S \subset M$ is a smooth hypersurface (without boundary that does not intersect ∂M). Suppose $\vec{N}$ is a section of TM restricted to S such that $\vec{N}$ is nowhere tangent to S. Then S is orientable with the orientation given by the form $(e_1, \ldots, e_{n-1}) \to \omega(\vec{N}, e_1, \ldots)$ (Indeed, $\vec{N}, e_1 \ldots$ are linearly independent and hence $\omega(\vec{N}, \ldots) \neq 0$.) For instance, S^n can be oriented this way.