MA 229/MA 235 - Lecture 2

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Recap

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- Constructed bump functions.

Review of multivariable calculus: Gradient and steepest ascent

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- For instance, if $f(x,y) = x^2 + 2y^2$, then at $(1,1)$, $Df_{(1,1)} = (2,4)$. Indeed, moving more in the y-direction would cause a greater increase in the height.

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$f_x = \frac{y(x^2-y^2)(x^2+y^2)+2x^2y(x^2+y^2)-2x^2y(x^2-y^2)}{(x^2+y^2)^2} = y\frac{x^4-y^4-4x^2y^2}{(x^2+y^2)^2}$,
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- Theorem (Clairaut) : If all the second partials exist and are continuous are a, then the mixed partials are equal.

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- Now, $w(h,k) = u(h,k) - u(0,k) = h\partial_x u(c_1, k)$ by MVT for $c_1 \in [0, h]$. This equals
  $h(\partial_x f(a+c_1, b+k) - \partial_x f(a+c_1, b)) = hk\partial_y \partial_x f(a+c_1, b+c_2)$
  where $c_2 \in [0, k]$.

- We will prove it for two-variable functions. The general case follows from this special case (why?)
- Let $u(h,k) = f(a+h, b+k) - f(a+h, b)$,
  $v(h,k) = f(a+h, b+k) - f(a, b+k)$, and
  $w(h,k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) - f(a, b)$.
- Now, $w(h,k) = u(h,k) - u(0,k) = h\partial_x u(c_1, k)$ by MVT for $c_1 \in [0,h]$. This equals
  $h(\partial_x f(a+c_1, b+k) - \partial_x f(a+c_1, b)) = hk\partial_y \partial_x f(a+c_1, b+c_2)$
  where $c_2 \in [0,k]$.
- Likewise,
  $w(h,k) = v(h,k) - v(h,0) = hk\partial_x \partial_y f(a+d_1, b+d_2)$ where
  $d_1 \in [0,h], d_2 \in [0,k]$.

Review of multivariable calculus: Proof of Clairaut

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- Using the C^2 Clairaut, we can prove the C^k Clairaut for any k.

Review of multivariable calculus: Proof of a version of Taylor's theorem

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Review of multivariable calculus: Proof of a version of Taylor's theorem

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Review of multivariable calculus: Proof of a version of Taylor's theorem

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Review of multivariable calculus: Proof of a version of Taylor's theorem

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Review of multivariable calculus: Proof of a version of Taylor's theorem

- Thus, $g_v(t) = \sum_{i=0}^{k} \sum_{|\alpha|=i} \frac{\partial_\alpha f(a)}{\alpha!} v^\alpha \frac{t^i}{i!} + h_v(t)$. Let $x - a = tv$. Then $f(x) = \sum_{i=0}^{k} \sum_{|\alpha|=i} \frac{\partial_\alpha f(a)}{\alpha!}(x-a)^\alpha + R(x,a)$ where $R(x,a) =$
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- The form stated in the previous lecture (the Peano form of the remainder) can be obtained by using L'Hopital's rule just as in the one-variable case.

Review of multivariable calculus: The second derivative test

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- Proof: Taylor's theorem (with the Peano form of the remainder) implies that if $\sum_{i,j} f_{ij}(a)v^i v^j > 0$,

Review of multivariable calculus: The second derivative test

- Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$ be a C^1 function. Suppose f is C^2 at $a \in U$ and $Df_a = 0$.
- Theorem: If $\sum_{i,j} f_{ij}(a)v^i v^j > 0$ for all $v \neq 0$, then a is a local minimum. If $\sum_{i,j} f_{ij}(a)v^i v^j < 0$ for all $v \neq 0$, then a is a local maximum. Conversely, if a is a local minimum $\sum_{i,j} f_{ij}(a)v^i v^j \geq 0$ for all v (and likewise for a local max).
- Proof: Taylor's theorem (with the Peano form of the remainder) implies that if $\sum_{i,j} f_{ij}(a)v^i v^j > 0$, $f(a+v) - f(a) > 0$ for all small v.

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