MA 229/MA 235 - Lecture 11

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Recap

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- Tangent spaces on manifolds. Pushforwards.

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- Tangent spaces on manifolds. Pushforwards.
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- Coordinate bases and pushforwards in terms of coordinates.

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- Let $\gamma : J \to M$ be a smooth map. Then γ is an immersion iff $\gamma'(t) \neq 0$ for all $t \in J$.
- A circle rotated about an axis can be thought of as an immersion of $\mathbb{R}^2$ into $\mathbb{R}^3$.
- A $1 - 1$ immersion need *not*

Examples and non-examples

- $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is *not* of constant rank. It is an immersion (and a submersion) at $x = 1$ for instance.
- $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x$ is a submersion. Likewise for projections from products of manifolds.
- $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by $f(x, y) = (x, y, 0)$ is an immersion. Likewise for inclusions into products of manifolds.
- Let $\gamma : J \to M$ be a smooth map. Then γ is an immersion iff $\gamma'(t) \neq 0$ for all $t \in J$.
- A circle rotated about an axis can be thought of as an immersion of $\mathbb{R}^2$ into $\mathbb{R}^3$.
- A $1 - 1$ immersion need *not* be a homeomorphism to its image.