HW 2 (to be submitted by Jan 28)

1. (7 marks) Consider the function $f(x^1, \ldots, x^n) = (x^1 x^2 \ldots x^n)^{1/n}$ defined on $x_i \geq 0 \ \forall\ i$.

   (a) (5 marks) Find the maximum value of f subject to $x^1 + x^2 + \ldots + x^n = c$.

   (b) (2 marks) Prove that AM-GM inequality.

2. (8 marks) This problem aims to generalise the Lagrange multipliers method to more than one constraint.

   (a) (3 marks) Suppose A is an $m \times n$ matrix of real numbers. Prove that A has full rank if and only if there exists an $r \times r$ invertible minor where $r = \min(m, n)$. (That is, we can permute the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ so that the first $r \times r$ block of the resulting new matrix is invertible.)

   (b) (2 marks) Let $U \subset \mathbb{R}^n$ be an open set and $G : U \to \mathbb{R}^m$ given by $G(x^1, \ldots, x^n) = (g^1(x), g^2(x), \ldots, g^m(x))$ be a smooth function ($m < n$). Suppose DG_a has full rank where $a \in U$. Prove that (make the following statement precise as a part of the exercise) near a, one can solve for m variables in terms of the other $n - m$ variables smoothly.

   (c) (3 marks) Let $f : U \to \mathbb{R}$ be a smooth function that attains a local extremum subject to $G = 0$ at $a \in U$. Assume that DG_a has full rank. Prove that there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such that $\nabla f(a) = \lambda_1 \nabla g_1(a) + \lambda_2 \nabla g_2(a) + \ldots$.

3. (5 marks) Prove that if a Hausdorff second-countable space such that every point p has a neighbourhood U_p that is homeomorphic to $\mathbb{R}^{n_p}$ is connected, then n_p is independent of p.

4. (5 marks) Prove that topological manifolds are metrizable.