HW 4 (to be submitted by Feb 18)

1. (15 marks) Let M be a manifold (with or without boundary). Consider the set $\mathcal{S}$ of all the coordinate charts (U, x) containing p. For every $(U, x) \in \mathcal{S}$, consider the vector space $V_{U,x} = \mathbb{R}^n$, i.e., consider the disjoint union $W = \cup_{U,x} V_{U,x}$ of $\mathbb{R}^n$ over U, x. Define a relation $\sim$ on W as $v \in V_{U,x} \sim w \in V_{W,y}$ if $v^i = w^j \frac{\partial x^i}{\partial y^j}(p)$.

   (a) Prove that this relation is an equivalence relation.

   (b) The set of equivalence classes is defined to be $\tilde{T_p M}$. Prove that it is a vector space under $[v] + [w] = [v + w]$ and $c.[v] = [cv]$.

   (c) Suppose $F : M \to N$ is a smooth map, define $\tilde{F}_*([v]) = [DFv]$. Prove that this definition is well-defined.

   (d) Consider the (choice-free/canonical) map $F : T_p M \to \tilde{T_p M}$ given by $v \to [v^i]$. Prove that this map is well-defined and a linear isomorphism that commutes with pushforwards. (Part of the exercise is to make sense of this statement.)

2. (10 marks) Prove that the closed unit ball in $\mathbb{R}^{n+1}$ is a smooth manifold-with-boundary and that the inclusion map is smooth.