1. Recap

(1) Proved that $X \subset \mathbb{R}$ is connected if and only if it satisfies the intermediate value property.
(2) As a corollary, proved several spaces (like the circle) are connected.
(3) Defined path connectedness and show that it implied connectedness.
(4) Showed that $\mathbb{R}^n - \{0\}$, the sphere, etc are connected.
(5) Showed that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}$ when $n > 1$.

However, the converse is NOT true in general. (Basically, you might need an "infinitely" long path in some sense.)

(1) The deleted Comb space : Let $K = \{\frac{1}{n}, \ n \in \mathbb{Z}_+\}$. Let C be $[0, 1] \times \{0\} \cup K \times [0, 1] \cup \{0\} \times [0, 1]$. Call C the comb space. Let $D \subset C$ (the deleted Comb space) be $C - \{0\} \times (0, 1)$. D is seen to be connected. Indeed it is a union of T-shaped connected sets (which we call X) and the point $p = \{1\} \times [0, 1]$. If $D = A \cup B$, then $X \subset A$ or $X \subset B$. Clearly $A = X$ and $B = p$. But p is not open in D. Indeed, it has points arbitrarily close to it. (Alternatively, p is contained in the closure of D.)
Suppose there is a path $f : [a, b] \to D$ joining p to a point in X. (X itself is path connected.) We claim that $S = f^{-1}(p)$ is open and closed in $[a, b]$ and hence equal to $[a, b]$ thus producing a contradiction.
    Indeed, it is obvious that S is closed by continuity. Suppose $r \in S$. Take any neighbourhood V of p that does not intersect the x-axis. Choose an open interval $U = (r - \epsilon, r + \epsilon)$ such that $f(U) \subset V$. Now $f(U)$ is connected. Suppose $q = (\frac{1}{n}, t_0)$ is a point in $f(U)$ other than p. Then choose r so that $\frac{1}{n+1} < r < \frac{1}{n}$. Then $(-\infty, r) \times \mathbb{R} \cap f(U) \cup (r, \infty) \times \mathbb{R} \cap f(U)$ is a separation of $f(U)$. This is a contradiction.
(2) Topologist's sine curve : Let $S = \{(x, \sin(1/x)), \ 0 < x \leq 1\}$. This is connected. Its closure $\tilde{S}$ (the topologist's sine curve) is also connected. It is not path connected. This is an exercise.

Given a space X, even it is disconnected, one can break it into a collection of connected pieces (called components)
Definition : $x \sim y$ if there exists a connected subspace $Y \subset X$ containing x and y. The equivalence classes are called connected components of X.
(Proof that it is an equivalence relation : Reflexivity and symmetry are obvious. If x, y are in a connected set A, and $y, z \subset B$, then $A \cup B$ is connected because $y \in A \cap B$.)

Another way is by the following theorem ("maximally connected sets are components").

Theorem 1.1. *The components of X are connected disjoint closed subsets whose union is all of X and each nonempty connected subspace is contained in one of them (and hence intersects exactly one of them).*

Proof. They are of course disjoint and their union is X. Suppose $Y \subset X$ is connected. Assume $Y \cap C \neq \phi$ where C is a component. Suppose $x \in Y \cap C$ and $y \in Y \cap C^c$. Then since $x, y \in Y$, $x, y \in C$ which is a contradiction unless $Y \subset C$.
Also, each component is connected. Indeed, if $C = A \cup B$ is a separation, then $x \in A$ and $y \in B$ cannot

be in a connnected subspace.

They are closed because $\bar{C}$ is also connected and hence contained in C.   □

Unfortunately, components need not be open. For example, $\mathbb{Q}$ is disconnected and every rational is a component. Indeed, if x, y are in a connected subspace Y, then consider an irrational $x < i < y$. Now $Y = (-\infty, i) \cap Y \cup (i, \infty) \cap Y$ is a separation. These are not open.

Another similar definition : $x \sim y$ if x and y can be joined by a path. The equivalence classes are called path components.

Proof of equivalence : x is connected to itself by the constant path (Reflexivity). In the path we may assume that $a = 0$ and $b = 1$ by rescaling. If $p(t) : [0, 1] \rightarrow X$ connects x, y, then $p(1 - t)$ connects y and x (Symmetry). If $p_1(t)$ connects x, y and $p_2(t)$ connects y, z, by rescaling, assume that $p_2(s) : [1, 2] \rightarrow X$. Then one can combine these two paths using the pasting lemma (Transitivity).

Akin to connectedness, we have the following theorem (with a small twist that we cannot say that path components are closed).

Theorem 1.2. *The path connected components are path connected disjoint subsets whose union is X and every path connected set intersects only one of them.*

Proof. This proof is similar to that for connectedness and is left as a HW problem.   □