NOTES FOR 18 SEPT (TUESDAY)

1. Recap

1. Discussed the deleted comb space and the topologist's sine curve.
2. Defined connected components and proved that they are the largest connected subsets (and that they are closed).
3. Defined path connected components and stated a similar result.

2. Components and local connectedness

If you take the topologist's sine curve, it has only one connected component but 2 path connected components (the vertical line S and the curve $\tilde{S}$ $y = \sin(1/x)$). Now $\tilde{S}$ is obviously open (why ?) in the space and hence its complement S is closed in the space. But $\tilde{S}$ is not closed and neither is S open. If one removes some points from S (namely the ones having rational y-coordinate), then the resulting space is still connected (because it is between $\tilde{S}$ and its closure) but it has uncountably many path components.

Now we deal with local properties (and the important question of when path connected spaces are connected).
Definition : A space is said to be locally connected at x if for every neighbourhood U of x, there is a connected neighbourhood $x \in V \subset U$. Likewise for locally path connected.

The real line is (locally) path connected. However, $(-1, 1) \cup (2, 3)$ is not connected but is locally connected. The topologist's sine curve is connected but not locally path connected. The rationals are neither connected nor locally so.

Theorem 2.1. *A space X is locally connected if and only if for every open set $U \subset X$, each component of U is open in X.*

Proof. If X is locally connected, then suppose $C \subset U$ is a component of U. Then if $x \in C$, by local connectedness, $x \in V \cap U$ is an open subset that is connected and hence $V \subset C$. This means that C is open in U and hence in X.
Conversely, suppose $x \in U \subset X$. Consider the component $V \subset U$ containing x. Since V is open in X and connected in U, this means that it is connected in X. □

Likewise, one can prove that a space is locally path connected if and only if every path component is open. Since the complement of a path component is a union of path components (each of which is open), each path component is also closed in this case.

Finally, we have a relationship between path components and components.

Theorem 2.2. *Each path component of X lies in a component. If X is locally path connected, they coincide.*

Proof. A path component C is path connected and hence connected. Therefore it lies in a component. Suppose C is a component. Fix a point $x \in C$. Let $P \subset C$ be the path component containing x. By local path connectedness, P, C are clopen in X. Now $C = P \cup (P^c \cap C)$ is a separation. □

3. Compactness

One of the aims is to find a condition on X to generalise the extreme value theorem. Originally it was thought that "every sequence has a convergent subsequence" should be the correct property. Unfortunately, this is not general enough. By trial and error one comes up with the following (non-intuitive) definition.

Definition 3.1. A collection of subsets is said to *cover* X if the union of all its elements is X. If these subsets are open, such a cover is called an *open cover*. A collection of subsets of X is said to cover a *subspace* Y if their union *contains* Y.
A topological space X is said to be compact if every open cover has a finite subcover, i.e., a finite subcollection of open sets that continue to cover X.

Examples and non-examples :

(1) Obviously any finite topology, i.e., having finitely many open sets is compact.
(2) The real line is not compact because the sets $(n-1, n+1)$ cover it but cannot have a finite subcover.
(3) The subspace $X = \{0\} \cup \{\frac{1}{n}\}$ where $n \in \mathbb{Z}_+$ is compact. Indeed, given an open cover of X, there is an element U containing 0. Hence $(-\frac{1}{N}, \frac{1}{N}) \subset U$ for some large N. Thus, all but finitely many $\frac{1}{n}$ are in U. The remaining finitely many elements are in finitely many open sets.
(4) The interval $(0, 1]$ is not compact. Indeed, take the open cover $(\frac{1}{n}, 1]$. This cannot have a finite subcover. Likewise, neither is $(0, 1)$. But it turns out that $[0, 1]$ is compact. We shall prove that soon.

The notion of compactness is a topological notion that behaves well with the subspace topology.

Theorem 3.2. *A subspace Y of X is compact in the subspace topology if and only if for every open cover of Y in X (every collection of open subsets of X whose union contains Y) there is a finite subcollection of open subsets of X whose union contains Y.*

Proof. IF : Suppose $\mathcal{A}$ is a collection of open subsets of Y that cover Y. Then every element of $\mathcal{A}$ is of the form $U \cap A$ where U is an open subset of X. The collection of U covers Y in X. Hence $Y \subset U_1 \cup U_2 \ldots U_n$. Thus, $Y = (U_1 \cap Y) \cup (U_2 \cap Y) \ldots$.
ONLY IF : If $Y \subset \cup_{i \in I} U_i$, where U_i are open in X, then let $\mathcal{A}$ consist of $U_i \cap Y$. By compactness, wlog, $Y = (U_1 \cap Y) \cup (U_2 \cap Y) \ldots (U_n \cap Y)$. Therefore, $Y \subset U_1 \cup U_2 \ldots U_n$. □

Now we have a couple of theorems relating closedness and compactness.

Lemma 3.3. *A closed subset A of a compact set X is compact.*

Proof. Suppose $A \subset \cup_{i \in I} U_i$. Then consider the open cover of X given by U_i and A^c. Since X is compact, a finite subcover exists, which obviously covers A as well. □

More non-trivially,

Theorem 3.4. *Compact subsets A of Hausdorff spaces X are closed.*

Proof. Suppose $x \in A^c$. We need to show that there is an open neighbourhood $x \in U \subset A^c$. We shall show something stronger than this fact.

Lemma 3.5. *If $A \subset X$ is compact, and X is Hausdorff, then for every $x \in A^c$, there exist two disjoint open subsets $U, V \subset X$ such that $x \in U$ and $A \subset V$.*

Proof. For every $a \in A$, by Hausdorffness, there exist disjoint open neighbourhoods $a \in V_a, x \in U_a \subset X$. Now, $A \subset \cup_a V_a$. Therefore, there are finitely many V_{a_i} covering A. Let $U = \cap_i U_{a_i}$. U is a finite intersection and hence open. $\cup_i V_{a_i} = V$ is open. Obviously, $U \cap V = \phi$. □

□

The Hausdorff condition is important as is seen in $X = \{a, b, c\}$ with the topology being $\tau = \{\phi, X, \{a, b\}$.
Once we prove that $[a, b] \in \mathbb{R}$ is compact, we see that any closed subspace of it is. Also, (a, b) and $(a, b]$ are not compact because they are not closed subsets of the Hausdorff space $\mathbb{R}$.
Just as for connected sets,

Proposition 3.6. *The image of a compact space X under a continuous function $f : X \to Y$ is compact.*

Proof. Left as an exercise. □