NOTES FOR 20 SEPT (THURSDAY)

1. Recap

(1) Prove that X is locally connected if and only if for every open set U, every component is open.
(2) Also proved that connected locally path connected sets are path connected.
(3) Defined compactness and gave examples and counterexamples.
(4) Stated that continuous functions take compact sets to compact sets, closed subsets of compacts are compact, and compact subsets of Hausdorff spaces are closed.

2. Compactness

Theorem 2.1. *Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof. Note that bijective closed continuous maps are homoeomorphisms. Now if $A \subset X$ is closed, it is compact. Thus $f(A) \subset Y$ is compact and hence closed. □

The hypotheses above are quite important. For instance,
Suppose X is finite. Then $Id : (X, discrete) \to (X, indiscrete)$ is bijective and continuous and X is compact (under both topologies) but not Hausdorff under the indiscrete topology. Of course Id cannot be a homeomorphism because the number of open sets don't tally.

Theorem 2.2. *Finite products of compact sets are compact.*

Actually, we have Tychonoff's theorem - Arbitrary products under the product topology of compact sets is compact. But this is fairly non-trivial (and uses a version of the axiom of choice). Let us prove only the finite version here. In fact, we shall prove that $X \times Y$ is compact when X and Y are. Finite products follows from induction.

Proof. We first prove a "tube lemma".

Lemma 2.3. *Consider the produce space $X \times Y$ where Y is compact. If $N \subset X \times Y$ is an open set containing a slice $S = \{x_0\} \times Y$, then N contains a "tube" $W \times Y$ where W is a neighbourhood of $\{x_0\}$.*

Proof. For every point $(x_0, y) \in S$, there exists a basis neighbourhood $(x_0, y) \in W_y \times U_y \subset N$. Now $\cup_{y\in Y} U_y = Y$ and hence by compactness of Y, $U_{y_1} \cup U_{y_2} \ldots U_{y_n} = Y$. Let $W = W_{y_1} \cap W_{y_2} \cap \ldots W_{y_n}$ be an open subset of X. Then $\cup_i W \times U_{y_i} = W \times Y \subset N$ is the desired tube. □

Now if $\mathcal{A}$ is an open cover of $X \times Y$, for every $x \in X$, consider the compact slice $S_x = \{x\} \times Y$. Hence, there is a finite subcover $V_{x,1} \ldots V_{x,n_x}$ from $\mathcal{A}$ covering this slice. Let $N_x = V_{x,1} \cup \ldots$. Now the slice $S_x \subset N_x$. By the tube lemma, there exists a tube $W_x \times Y \subset N_x$. The open sets W_x cover the compact set X and hence there is a finite subcover $W_{x_1}, \ldots, W_{x_m}$. Thus $V_{x_i,j}$ is a finite subcover of $X \times Y$. □

It is crucial for the tube lemma that Y be compact. Here is a counterexample - $N = \{(x, y) \in \mathbb{R}^2 : |x| < \frac{1}{y^2+1}\}$. This has no tube of around $\{0\} \times \mathbb{R}$.

Recall the nested intervals theorem in real analysis - If you have a sequence of nested closed bounded intervals, their intersection is non-empty. (This is not true if they are open for instance. Ex - $(0, 1) \supset (\frac{1}{2}, 1) \supset (1 - \frac{1}{3}, 1) \ldots$.)

Actually compactness is the crucial generalisation of this theorem. (The following theorem is used in the proof of the Tychonoff and the Baire category theorems.) Indeed,

Theorem 2.4. *Let X be a topological space. Then X is compact if and only if for every collection of closed sets in X having the finite intersection property (that is, every finite intersection is non-empty), the intersection of all the sets is non-empty.*

Proof. Suppose X is compact : Take the complements of the closed sets $U_i = C_i^c$. If $\cap_i C_i = \phi$, then $\cup_i U_i = X$ (by De Morgan's laws). Then there is a finite subcover $X = U_1 \cup U_2 \ldots U_n$. This means that $C_1 \cap C_2 \ldots C_n = \phi$. A contradiction.
Conversely, if U_i is an open cover of X without a finite subcover then $U_i^c = C_i$ are closed sets having the finite intersection property. Hence $\cap_i C_i \neq \phi$ which means that U_i cannot be an open cover. □

3. Compact subsets of the real line

Firstly,

Lemma 3.1. *Closed bounded intervals $[a, b] \subset \mathbb{R}$ are compact.*

Proof. Two proofs. Suppose U_i form a cover of $[a, b]$. Then :

(1) The set S of all numbers $b \geq c \geq a$ such that $[a, c]$ is covered by a finite subcollection is bounded above and non-empty. Hence it has a least upper bound l. Now $l \in U_j$ for some j. Thus $(l - \epsilon, l + \epsilon) \subset U_j$ for some $\epsilon > 0$. Since l is the l.u.b of S, there exists a $c \in (l - \epsilon, l) \cap S$. This means that $[a, l + \frac{\epsilon}{2}]$ can be covered by a finite subcollection (including U_j if necessary).
(2) Take the set S as above. It is non-empty. We will show that it is open and closed. Hence, by the connectedness of $[a, b]$, $S = [a, b]$. If $c \in S$, then $c \in U_j$ for some j. So $(c - \epsilon, c + \epsilon) \subset U_j$. This means that $[a, c + \frac{\epsilon}{2}]$ can be covered by a finite subcollection (including U_j). This shows openness.
To show that S is closed, suppose $c \in S^c$. So $c \in U_k$ for some k, meaning that so is $(c - \delta, c + \delta) \subset U_k$. If any element of $(c - \delta, c + \delta)$ is in S, then the entire open interval is in S (by simply including U_k in the finite subcollection). Hence S^c is open.

□

This can be used to prove the following important result.

Theorem 3.2. *A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and bounded under the Euclidean metric.*

Proof. Note that the Euclidean metric is equivalent to the "square" metric. Hence a subset is bounded if and only if it is contained in a large closed cube.
Suppose A is closed and bounded : Since $[a, b]$ is compact, so is any closed cube (it is a Cartesian product of closed intervals). Since A is bounded, it is a subset of a large cube, which is compact. So A is compact.
Suppose A is compact : Since $\mathbb{R}^n$ is Hausdorff, A is closed. Suppose A is not contained in any cube, i.e., if one takes a sequence of open cubes C_n of size n centred at the origin, then there is at least one

point $p_n \in A$ outside C_n for every n. So there cannot be a finite subcover of this open cover of A. (This is an open cover of all of $\mathbb{R}^n$ actually.) □

(Warning : Please note that this property is true for $\mathbb{R}^n$ but may fail for other metric spaces. For instance, if you take the space $C[a, b]$, then closed bounded sets need not be compact. The missing condition is called equicontinuity. This is the content of the Arzela-Ascoli theorem.)