NOTES FOR 4 OCT (THURSDAY)

1. Recap

(1) Proved that finite products of compact sets are compact.
(2) A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. (This is not true for other metric spaces.)

2. Compactness

In the exam we proved that

Theorem 2.1. *Compact subsets of metric spaces are closed and bounded.*

Now we prove the following generalisation of the extreme value theorem.

Theorem 2.2. *If $f : X \to \mathbb{R}$ is a continuous function and X a compact space, then there exist $a, b \in X$ such that $f(a) \geq f(x)\ \forall\ x \in X$ and likewise $f(b) \leq f(X)\ \forall\ x \in X$.*

Proof. f takes compact sets to compact sets. Hence $f(X) \subset \mathbb{R}$ is closed and bounded. Suppose $M = \sup_{x \in X} f(x)$. (Of course $M < \infty$ because $f(X)$ is bounded.) Assume that $M \notin f(X)$. Since $f(X)$ is closed, there exists an interval $(M - \epsilon, M + \epsilon) \notin f(X)$ (because the complement is open). This contradicts the fact that M is the least upper bound. Likewise, $\inf f(X) \in f(X)$. □

Now we make a definition - Suppose $A \subset X$ is nonempty, X is a metric space, and $x \in X$. The distance to A is defined as $d(x, A) = \inf\{d(x, a)\ :\ a \in A\}$. It is not at all obvious that $d(x, A) = d(x, a_x)$ for some $a_x \in A$. In fact, this is usually not true. For instance, take $x = 0$ and $A = (0, 1)$. Then $d(x, A) = 0$ but $x \notin A$.
Here is the first observation about this distance function :

Lemma 2.3. *The function $f(x) = d(x, A)$ is a continuous function of x.*

Proof. Suppose $x, yinX$. By definition, $d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a)$ for every $a \in A$. Hence, $d(x, A) - d(x, y) \leq d(y, a)$ and hence $d(x, A) - d(x, y) \leq d(y, A)$. This means that $d(x, A) - d(y, A) \leq d(x, y)$ and the same inequality holds with x and y interchanged. Hence continuity holds. □

Recall that the diameter of a subset $A \subset X$ of a metric space is $\sup_{a,b \in A} d(a, b)$. We prove a very important lemma - the Lebesgue Number Lemma (especially in algebraic topology) about the diameter.

Theorem 2.4. *Let $\mathcal{A}$ be an open cover of the metric space (X, d). If X is compact, there exists a number $\delta_{\mathcal{A}}$ (called the Lebesgue Number of the open cover $\mathcal{A}$) such that every subset of X having diameter $< \delta$ is contained in some element of $\mathcal{A}$.*

Proof. Assume without loss of generality that X itself is not an element of $\mathcal{A}$.
Choose a finite subcover $A_1, \ldots, A_n$ of X. For each i, let $C_i = X - A_i$. Define $f : X \to \mathbb{R}$ by $f(x) = \frac{\sum_i d(x, C_i)}{n}$.
Claim : $f(x) > 0$ for all $x \in X$. Proof : Given $x \in X$, choose an i so that $x \in A_i$. Since A_i is open, an ϵ-ball around x also lies in A_i. Hence, $d(x, C_i) \geq \frac{\epsilon}{n}$.

Since f is continuous on a compact set, it has a minimum value $\delta > 0$. We claim that $\delta_{\mathcal{A}} = \delta$. Indeed, suppose $B \subset X$ has diameter less than δ. Choose a point $x_0 \in B$. Clearly, the δ open ball around x_0 contains B. Now $\delta \leq f(x_0) \leq d(x_0, C_m)$ where $d(x_0, C_m)$ is the largest of all $d(x_0, C_i)$. Then the δ-neighbourhood of x_0 is contained in the element A_m. □

Now we make a definition : A function $f : (X, d_X) \to (Y, d_Y)$ is called uniformly continuous if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $d(x, y) < \delta$, $d(f(x), f(y)) < \epsilon$ (that is, δ depends only on ϵ and not on x, y).

Theorem 2.5. *If $f : (X, d_X) \to (Y, d_Y)$ is continuous and X is compact, then it is uniformly continuous.*

Proof. Given $\epsilon > 0$, take the open cover of Y given by the open balls $B(y_0, \frac{\epsilon}{2})$. Let $\mathcal{A}$ be the open cover of X given by the inverse images of these balls. Let δ be the Lebesgue number of this open cover. Then, whenever $d(x, y) < \delta$, x, y are in one of the A_i. Thus $d(f(x), y_i) < \frac{\epsilon}{2}$ and likewise for y. By the triangle inequality we are done. □

To go further, we shall define the notion of a limit point : x is said to be a limit point of a subset $A \subset X$ if every open neighbourhood of x intersects in A in a point other than x itself, i.e., every "deleted open neighbourhood intersects A". We have the following result.

Theorem 2.6. *Let $A \subset X$. Then $\bar{A}$ consists of A along with all of its limit points A's.*

Proof. Firstly,

Lemma 2.7. *$x \in \bar{A}$ if and only if every neighbourhood of x intersects A.*

Proof. If $x \notin \bar{A}$ then $U = X - \bar{A}$ is an open set that contains x not intersecting A.
If U is an open neighbourhood of x not intersecting A, then U^c is closed and contains A. Hence $\bar{A} \subset U^c$. This means that $x \notin \bar{A}$. □

Now, if $x \in A \cup A'$, by the above theorem, $x \in \bar{A}$. So $A \cup A' \subset \bar{A}$.
If $x \in \bar{A}$, then either $x \in A$ or every neighbourhood of x intersects A and hence x is a limit point of A. Thus $\bar{A} \subset A \cup A'$. □

For future reference, here is an important definition - A sequence $x_1, \ldots, x_n, \ldots$ is said to converge to x if for every neighbourhood U of x, there exists an N_U such that $n \geq N_U$ implies that $x_n \in U$. (Note that x is a limit point of the sequence.) It is easy to show that if X is Hausdorff, every sequence converges to at most one point.