NOTES FOR 4 SEPT (TUESDAY)

1. Recap

(1) Definition of a topology in terms of open (and equivalently, closed) sets. Examples (Indiscrete, discrete, metric, subspace, product, quotient).
(2) Quotient map examples : Open (projections are open but not necessarily closed) and closed maps ($[0,1] \cup [2,3] \rightarrow [0,2]$ by a piecewise linear map is closed but not open) and examples which are neither but quotient maps nonetheless.
(3) Connectedness - Motivation - Find a property that generalises the IVT (Why care about the IVT in the first place ?)
(4) Connectedness - Definition - A separation of X is a pair $U \neq \phi, V \neq \phi$ of disjoint open sets such that $X = U \cup V$. Clearly, U, V are clopen in X. X is connected if there is no such separation.
(5) Examples and counterexamples of connected sets
    (a) Discrete topology - No non-singleton is connected.
    (b) Indiscrete topology - All proper non-empty subsets are connected.
    (c) Rationals are disconnected ($(-\infty, \sqrt{2}) \cap \mathbb{Q}$ and its complement induce a separation).
(6) Theorem : X is connected if and only if every continuous function $f : X \rightarrow \{0, 1\}$ is a constant.
(7) Theorem : If $X = A \cup B$ is a separation, and $Y \subset X$ is connected, then $Y \subset A$ or $Y \subset B$.
(8) Corollaries :
    (a) If $A \subset X$ is connected, and $A \subset B \subset \bar{A}$, then B is connected.

    Proof. If $B = \bar{A}$, then B is closed. Suppose B is disconnected. Then there is a separation $B = U \cup V$ where U, V are clopen in B. Hence they are closed in X. Since A is connected, either $A \subset U$ or $A \subset V$ meaning that $\bar{A} = B \subset U$ or $B \subset V$. Hence we are done.
    In the general case, suppose B is disconnected and $B = U \cup V$. Then $A \subset U$ or $A \subset V$. Hence, $\bar{A} \subset \bar{U}$ or $\bar{A} \subset \bar{V}$ because $\bar{A}$ was proven to be connected. Suppose $\bar{A} \subset \bar{U}$. Then $B \subset \bar{U}$ which means that $\bar{U} \cap V \neq \phi$. But since U, V are closed in B, $\bar{U} \cap B$ (which is the closure of U in B) equals U and hence $\bar{U} \cap V = \phi$ which is a contradiction. □

    (b) If $C_i \subset X$ are connected sets such that $C_i \cap C_j \neq \phi \ \forall\, i, j$, then $\cup_i C_i$ is connected.

    Proof. Suppose $A \cup B = \cup_i C_i$ is a separation. Then by the above theorem, $C_i \subset A$ or $C_i \subset B$ for every i. By assumption there are i, j so that $C_i \subset A$ and $C_j \subset B$. This is a contradiction. □

2. Connectedness

Theorem : If $f : X \rightarrow Y$ is continuous and surjective, then Y is connected if X is so.

Proof. If $Y = U \cup V$ is a separation, then $f^{-1}(U), f^{-1}(V)$ are open. So, $X = f^{-1}(Y) = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$. Now $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \phi$ (because f is onto). □

Theorem : $X \times Y$ is connected if X and Y are so.
Corollary : $X_1 \times X_2 \ldots X_n$ is connected if X_i are so. (This follows from induction.)

Proof. There are two proofs of this result :

(1) Fix a point $(a, b) \in X \times Y$. Now the "lines" $x \times Y$ and $X \times b$ intersect at (x, b). Each of these is connected (because they are homeomorphic to Y and X respectively). By the above theorem, $T_x = x \times Y \cup X \times b$ is connected. Note that $(a, b) \in T_x$. Thus, $\cup_x T_x$ is connected. But this is $X \times Y$.

(2) Suppose $X = U \cup V$ is a separation. Then since $x \times Y \simeq Y$ is connected, $x \times Y \subset U$ or $x \times Y \subset V$ for all $x \in X$. Let A_x be the set of all x such that $x \times Y \subset U$ and likewise B_x for V. Since U is open, if $(x, y) \in U$, there is a neighbourhood $U_{x,y} \subset U$. Thus, $H_x = \cup_y U_{x,y} \subset U$ is open. Now $A_x = \pi_1(\cup_x H_x)$ and since π_1 is an open map, A_x is open. Likewise, so is B_x. But $A_x \cap B_x = \phi$. This is a contradiction.

□

3. Interlude - Product and Box topologies

On infinite products $\Pi_\alpha X_\alpha$, there are two kinds of topologies -
The Box topology - It is generated by a basis of the form $\Pi_\alpha U_\alpha$ where $U_\alpha \subset X_\alpha$ are arbitrary open sets.
The Product topology (the more useful one) - Generated by a basis of the above form where all but finitely many U_α are equal to X_α.

Theorem : If $f : A \to X = \Pi_{\alpha \in J} X_\alpha$ is a function $f(a) = (f_\alpha(a))_{\alpha \in J}$ where $f_\alpha : A \to X_\alpha$ are functions, then if X has the product topology, f is continuous iff f_α are so. This is not necessarily the case if X has the box topology.

Proof. Indeed, let $X = \mathbb{R}^\omega$. Then $\mathbb{R} \to X$ given by $t \to (t, t, \ldots)$ is not continuous w.r.t to the box topology. Take $(-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \ldots$. The inverse image is the non-open set $\{0\}$.

The continuity w.r.t the product topology is left as an exercise. □

Theorem : If X_α are connected, then $\Pi_\alpha X_\alpha$ is so in the product topology but not necessarily so in the box topology.