NOTES FOR 6 SEPT (THURSDAY)

1. Recap

(1) Recalled basics with examples. Recalled the definition of connectedness and provided examples and counterexamples.
(2) Theorem : If $X = A \cup B$ is a separation, and $Y \subset X$ is connected, then $Y \subset A$ or $Y \subset B$.
(3) Corollaries :
  (a) If $A \subset X$ is connected, and $A \subset B \subset \bar{A}$, then B is connected.
  (b) If $C_i \subset X$ are connected sets such that $C_i \cap C_j \neq \phi \ \forall \ i, j$, then $\cup_i C_i$ is connected.
(4) Continuous functions map connected sets to connected sets.
(5) $X \times Y$ is connected if X, Y are so.
(6) The most obvious topology on $X = \Pi_\alpha X_\alpha$ is the box topology and is badly behaved (product of connected sets needs not be connected and $f : A \to X$ need not be continuous even if its individual components are so). The correct product topology is generated by a basis $\Pi_\alpha U_\alpha$ where U_α is X_α for all but finitely many α.

2. Connected subsets of the real line

Theorem : $X \subset \mathbb{R}$ is connected if and only if for every $a < b \in X$, $[a, b] \subset X$.

Firstly, we prove that (a, b) is connected.

Proof. Suppose there is a separation $(a, b) = U \cup V$ where $U, V \neq \phi$ are open in (a, b) (and hence in $\mathbb{R}$). Let $a < c \in U$ and $b > d \neq c \in V$. Wlog assume that $c < d$. Then consider the set A of all $r \in (c, d)$ such that $(c, r) \in U$. Now $A \neq \phi$ because U is open and $c \in U$. It is bounded above and hence has a least upper bound l. Now $l \notin U$ because if it is so, then $l + \epsilon \in U$ for some small ϵ. So $l \in V$. But $l - \epsilon \notin V$ for any small ϵ. This is a contradiction because V is open. □

As a corollary, $(a, b) \subset B \subset [a, b]$ are all connected. In fact, a, b can be $\pm\infty$ too. (Hence $\mathbb{R}$ is connected.) Also, the circle is connected because it is $x \in \mathbb{R} \to (\cos(x), \sin(x))$.

Now we prove the main theorem.

Proof. IF : Suppose X is disconnected and $X = U \cup V$ is a separation into open subsets where $U = A \cap X$, $V = B \cap X$ where A, B are open in $\mathbb{R}$. Since $U, V \neq \phi$, there exists $a \in U$ and $b \in V$ where $a < b$ wlog. By assumption, $[a, b] \in X$. Thus $[a, b] \in U$ or $[a, b] \in V$ which is a contradiction.
ONLY IF : Suppose there exist $a < c < b \in X$ such that $a, b \in X$ but $c \notin X$. Then $X = X \cap (-\infty, c) \cup X \cap (c, \infty)$ is a separation. □

As a corollary, we see that the IVT holds.

3. Path connectedness

Definition : Given two points $x, y \in X$, a path f is a continuous map $f : [a, b] \to X$ such that $f(a) = x$ and $f(b) = y$. A space is called path connected if any two points can be joined by a path.

Lemma 3.1. *A path connected space is connected.*

Proof. Suppose $X = A \cup B$ is a separation of X. Then there exist $x \in A$ and $y \neq x \in B$. Let $f : [a, b] \to X$ be a path connecting them. The image of f is connected. Hence $f[a, b] \in A$ or in B. This is a contradiction. ◻

As a corollary, the unit disc is connected. (In fact, any convex subset is connected.)
$\mathbb{R}^n - \{0\}$ is connected.
The n-sphere is connected because it is the image of $\mathbb{R}^n - \{0\}$ under $\frac{x}{|x|}$.