NOTES FOR 9 OCT (TUESDAY)

1. Recap

(1) Generalised the extreme value theorem.
(2) Proved the Lebesgue Number Lemma.
(3) Defined uniform continuity and proved that continuous functions on compact metric spaces are uniformly so.
(4) Defined limit points and convergence of sequences.

2. Compactness

Now we give an interesting proof of the uncountability of $\mathbb{R}$. For this, define an isolated point $x \in X$ to be one such that $\{x\}$ is open in X.

Theorem 2.1. *If X is a nonempty compact Hausdorff space and X has no isolated points, then X is uncountable.*

Proof. There are two steps.

(1) Claim : Given any nonempty open set $U \subset X$ and any point $x \in X$, there is a nonempty open set $V \subset U$ such that $x \notin \bar{V}$.

   Proof. Choose a point $y \in U$ different from x. If $x \notin U$, then this obvious. If $x \in U$, use the fact that it is not isolated. Now choose disjoint open sets W_1, W_2 about x, y respectively. The set $V = W_2 \cap U$ is the desired set.
   Indeed, its closure does not contain x : Suppose it did. Then x is a limit point of V, which means that $W_1 \cap U$ intersects V - A contradiction. □

(2) Claim : Given $f : \mathbb{Z}_+ \to X$, the function f is not surjective.

   Proof. Let $x_n = f(n)$. Apply step 1 to $U = X$ to get a smaller V_1 such that $\bar{V}_1$ does not contain x_1. In general, given V_{n-1}, open and nonempty, choose V_n to be open and nonempty such that $V_n \subset V_{n-1}$ and $\bar{V}_n$ does not contain x_n. By the generalised nested intervals theorem, there exists $x \in \cap \bar{V}_n$. Of course $x \neq x_n$ for any n by construction. □

□

As a corollary, closed bounded intervals of $\mathbb{R}$ are uncountable. (Why does this mean that $\mathbb{R}$ is so ?)

3. Limit point compactness

A set X is defined to be limit point compact if every infinite subset of X has a limit point in X.

Lemma 3.1. *Compactness implies limit point compactness (but not vice-versa in general).*

Proof. If A is an infinite subset of the compact set X and A does not have a limit point, then A vacuously contains all its limit points. Hence A is closed. Furthermore, for every $a \in A$, there is an open set U_a intersecting A in a alone. Thus U_a, $X - A$ form an open cover of X. By compactness, only finitely many U_a cover A which means that A is finite. Contradiction.
Here is a counterexample to the converse : Let $Y = \{a, b\}$ with the indiscrete topology. The space $X = \mathbb{Z}_+ \times Y$ is limit point compact. (In fact every nonempty subset of X has a limit point. Indeed, if (n, a) is in the subset, then (n, b) is a limit point because every neighbourhood of (n, b) will contain (n, a).) But X is of course not compact. The open cover $\{n\} \times Y$ has no finite subcover. □

Definition : A set X is said to be sequentially compact if every sequence has a convergent subsequence.

Theorem 3.2. *For metric spaces X, TFAE.*

(1) *X is compact.*
(2) *X is limit point compact.*
(3) *X is sequentially compact.*

Proof. $1 \Rightarrow 2$ was done earlier.
$2 \Rightarrow 3$: If $A = \{x_1, \ldots, x_n, \ldots\}$ is a sequence, it has a limit point x if A is not finite. Now, consider the open balls $B_n = B(x, \frac{1}{n})$. Let $x_{n_1} \neq x$ be a point in $A \cap B_1$. We claim that given $x_{n_1}, \ldots, x_{n_k}$ such that $n_1 < n_2 < \ldots$ and $x_{n_i} \in B_i$, we can find an $n_{i+1} > n_i$ such that $x_{n_{i+1}} \in B_{i+1}$. This would imply that $x_{n_i} \to x$.
Indeed, this claim follows from the following lemma.

Lemma 3.3. *The ball B_i intersects A in infinitely many points.*

Proof. Suppose B_i intersects A in only finitely many points $a_1, a_2, \ldots, a_k$. Choose an integer $N > \max_i \frac{1}{d(x, a_i)}$. Then $B_N - \{x\} \cap A = \phi$. This is a contradiction. □

$3 \Rightarrow 1$: To prove this we need to reprove the Lebesgue number lemma for sequentially compact sets. Indeed,

Lemma 3.4. *If X is sequentially compact, and $\mathcal{A}$ is an open cover, then there exists a number $\delta_{\mathcal{A}} > 0$ (the Lebesgue Number of the cover) such that every set with diameter $< \delta_{\mathcal{A}}$ is in one of the elements of $\mathcal{A}$.*

Proof. Suppose there is no such δ. Then for every n, there is a set A_n with diameter $< \frac{1}{n}$ not completely contained in any element of $\mathcal{A}_n$. Choose $x_n \in A_n$. The sequence x_n has a convergent subsequence x_{n_i} (convergin to x). Since $x \in U$ for some $U \in \mathcal{A}$ and U is open, $B(x, \frac{1}{N}) \subset U$. Since $x_{n_i} \to x$, for all $i \geq M$, we see that $x_{n_i} \in B(x, \frac{1}{N})$. Hence for sufficiently large i, $A_{n_i} \subset U$, a contradiction. □

We need one more ingredient -

Lemma 3.5. *If the metric space X is sequentially compact, for every $\epsilon > 0$, there exists a finite cover by open ϵ-balls.*

Proof. Suppose not for some $\epsilon > 0$. Then choose any x_1. Since $B_1 = B(x_1, \epsilon) \neq X$, choose $x_2 \in B_1^c$. Since $B_1 \cup B_2 = B(x_2, \epsilon) \neq X$, choose x_3 not lying in these two and so on. Clearly $d(x_i, x_n + 1) \geq \epsilon \; \forall \, i = 1, \ldots, n$. Thus x_i can have no convergent subsequence. A contradiction. □

Now we complete the proof of $3 \Rightarrow 1$ (and hence the theorem). Let $\mathcal{A}$ be an open cover of X and δ its Lebesgue Number. Let $\epsilon < \delta$. Cover X with finitely many ϵ-balls. Each ball is in some element of $\mathcal{A}$. These finitely many elements cover X. □

4. Countability axioms

The above criterion of compactness applies more generally. To this end, we need "countability" axioms (so that we may sometimes check things on sequences).

(1) First countable : A space X is called first countable if for every point $x \in X$, there is a countable collection of neighbourhoods $\mathcal{B}_x$ such that any neighbourhood of x contains an element of $\mathcal{B}_x$. (Every point has a countable basis.)
(2) Second countable : A space X is called second countable if there is a countable basis for its topology. (That is a countable collection of open sets such that every open set is a union of these open sets.) Obviously second countability implies first countability.

Examples and counterexamples :

(1) The co-countable topology on $\mathbb{R}$ is not first countable. Indeed, the intersection of a countable neighbourhood local basis is still cocountable and hence open. Now delete one point (other than x) from this intersection to get something that does not contain any local basis element.
(2) All metric spaces are first countable.
(3) The metric space $\mathbb{R}_D$ with the discrete topology is first countable but not second countable. Indeed, the singletons $x\}$ are uncountably many open sets (and hence the countable basis cannot have all of them).
(4) $\mathbb{R}$ itself is second countable. Indeed, take $(q - \frac{1}{n}, q + \frac{1}{n})$ for all rationals q and positive integers n.

The point of first countability is that closedness can be checked using sequences.

Theorem 4.1. *Let X be a topological space.*

(1) *Let $A \subset X$. If there is a sequence of points $x_i \in A$converging to x, then $x \in \bar{A}$. The converse holds if X is first countable.*
(2) *Let $f : X \to Y$. If f is continuous, for every $x_n \to x$, $f(x_n) \to f(x)$. The converse holds if f is first countable.*

This will be given as a HW problem. Now both countability axioms behave well under the subspace and countable product operations.

Theorem 4.2. *A subspace of a first/second countable space is first/second countable. Likewise for countable products.*

Proof. We prove this for second countability. First countability is similar. If $\mathcal{B}$ is a countable basis for X, then $\mathcal{B} \cap A$ is a countable basis for A. (Why?) If $\mathcal{B}_i$ is a countable basis for X_i, then ΠU_i where all but finitely many U_i are X_i (and the rest are in $\mathcal{B}_i$) is a countable basis for $\Pi_i X_i$. □