MA 235 - Lecture 19

1 Recap

1. Diffeomorphism group acts transitively.
2. Vector fields as coordinate vector fields.
3. Lie brackets.

2 Cotangent bundle and one-forms

Recall that T^*M is set theoretically, $\cup_p T_p^*M$. T^*M is a vector bundle over M. Given a coordinate chart (U, x), a local smoothly varying basis for T^*M is given by $(\frac{\partial}{\partial x^i})^*$. A one-form is an element of T_p^*M. A one-form field is a collection of smoothly varying one-forms (a smooth section of T^*M), i.e., $\omega : M \to T^*M$ such that $\omega(p) \in T_p^*M$ and around every point, there exists a coordinate chart (U, x) such that $\omega = \sum_i \omega_i (\frac{\partial}{\partial x^i})^*$ where the functions ω_i are smooth. As before, if ω is smooth in one chart, it is so in all charts: $\tilde{\omega_i} = \omega_j \frac{\partial x^j}{\partial \tilde{x}^i}$ (why?). Moreover, given an atlas of U_α of M, and a collection of functions $\omega_{i,\alpha}$ such that on $U_\alpha \cap U_\beta$, $\omega_{i,\alpha} = \omega_{j,\beta} \frac{\partial x_\beta^j}{\partial x \alpha^i}$ then there exists a smooth one-form field ω on M whose coordinate representations are given by $\omega_{i,\alpha}$ (why?)

Given a smooth function $f : M \to \mathbb{R}$, what is its derivative? Sure, one way to answer the question is to say it is f_*. However, f_* can at best be thought of as a function $f_* : TM \to T\mathbb{R}$. On the other hand, if $M = \mathbb{R}^n$, we can think of the derivative as $Df : \mathbb{R}^n \to \mathbb{R}^n$. So is there a function $df : M \to$ *something* ? The most naive way to define it is take a chart (U, x) and take $(\frac{\partial f}{\partial x^1}, \ldots)$. However, when we change charts, these vectors change to $(\frac{\partial f}{\partial x^i} \frac{\partial x^i}{\partial \tilde{x}^1}, \ldots)$ which is exactly the way one-form fields change! Thus df must be thought of as a one-form field! Invariantly speaking, $df(X_p) := X_p(f)$. Suppose x^i are coordinates, then $dx^i(\frac{\partial}{\partial x^j}) = \delta_j^i$, i.e., $dx^i = (\frac{\partial}{\partial x^i})^*$. Thus any one-form field ω is $\omega = \omega_i dx^i$. Suppose γ is a smooth path, then $df(\gamma') = \gamma'(f) = (f \circ \gamma)'$. Thus in some sense, df is the "right" analogue of Newton's infinitesimals. $d(fg) = dfg + fdg, d(f+g) = df + dg, d(f/g) = \frac{dfg - fdg}{g^2}, d(c) = 0$.
Moreover, if $df \equiv 0$ on a connected manifold, f is constant on the manifold: Indeed, let $f(p) = c$ for some p. Then the set of all $q \in M$ such that $f(q) = p$ is non-empty and closed. It is also open: on a coordinate neighbourhood of q, $df = 0$ iff $\frac{\partial f}{\partial x^i} = 0$ and hence $f = c$ on that neighbourhood. By connectedness we are done. $\square$

Just as we can pushforward tangent vectors $(F_*)_p : T_pM \to T_{F(p)}N$, the dual map can be used to pullback cotangent vectors/one-forms: $(F^*)_{F(p)} : T^*_{F(p)}N \to T^*_pM$ given by $(F^*)_{F(p)}(\omega_{F(p)})(X_p) = \omega_{F(p)}((F_*)_pX_p)$. In fact, while we cannot pushforward vector fields, we can always pullback one-form fields: $(F^*)\omega(p)(X_p) = \omega_{F(p)}((F_*)_pX_p)$. In coordinates, $F*(dx^i)(\frac{\partial}{\partial x^j}) = dx^i(\frac{\partial F^k}{\partial x^j}\frac{\partial}{\partial x^k}) = \frac{\partial F^i}{\partial x^j}$, i.e, $F^*(dx^i) = dF^i = dF^i$. For ease of notation, if we denote $F^*f = f \circ F$, then $F^*(\omega_i dx^i) = \omega_i \circ F dF^i = F^*\omega_i dF^*x^i$. As an example, if $y = F(x) = x^2$ and $\omega = 3y^4dy$, then $F*\omega = 3(x^2)^4 2xdx$.

3 Tensors

If we want to measure *infinitesimal distances* on a manifold, we would need a "smoothly varying inner product". How does one define such an object? In physics, if we press an elastic body, how will it react? To know that, we would need to know a linear function that takes the normal to a surface and produces the "stress vector" across the surface. (The resulting linear map/matrix is called the stress tensor.) The area of a parallelogram is $\vec{a} \times \vec{b}$. The volume of a parallelopiped is $(\vec{a} \times \vec{b}).\vec{c}$. What about in higher dimensions? On a related note, how can one generalise the "cross product" to higher dimensions?
A common thread in all the questions above is the notion of a *multilinear* map or simply an object that has more than one index (like $A_{ijk\ldots}$). More so, we need a "smoothly varying family" of multilinear maps. Presumably, it corresponds to the section of some vector bundle.
Let V_i, W be vector spaces (over the same field). Then $T : V_1 \times \ldots V_k \to W$ is called multilinear if it is linear separately in each variable.
Examples: Dot product in $\mathbb{R}^n$, Cross product, Determinant, Lie bracket, etc.
Non-example: $T : \mathbb{R}^2 \to \mathbb{R}$ given by $T(x,y) = x + y$ is linear but not multilinear! $T(x,y) = xy$ is multilinear but not linear.
Another example: Let $\omega, \eta \in V^*$. Consider $\omega \otimes \eta : V \times V \to \mathbb{R}$ given by $\omega \otimes \eta(v,w) = \omega(v)\eta(w)$. This is a multilinear map. This example can be generalised to define the tensor product of arbitrary multilinear functionals. It is easily seen to be associative. Recursively, we can talk about $\omega_1 \otimes \omega_2 \ldots$. Likewise, since $V = V^{**}$ (in f.d), we can talk about $v \otimes w \otimes \ldots$.
Theorem: Let $n_i = dim(V_i)$. The dimension of the space $Mult(V_1, \ldots, V_k; \mathbb{R})$ is $n_1 n_2 \ldots$ and a basis is $(e_1^{i_1}) \otimes (e_2^{i_2}) \ldots$.
Proof: This set is linearly independent: Indeed, if $c_{i_1 i_2 \ldots}(e_1^{i_1}) \otimes (e_2^{i_2}) \ldots = 0$, then acting on $(e_{1,j_1}, e_{2,j_2}, \ldots)$ we get $c_{j_1 j_2 \ldots} = 0$. This is true for all $j_1, j_2 \ldots$. Hence we are done.
It spans the space: Let F be a multilinear functional. Define $F_{i_1 i_2 \ldots} = F(e_{1,i_1}, e_{2,i_2}, \ldots)$. Now consider $\omega = F_{i_1 i_2 \ldots}(e_1^{i_1}) \otimes (e_2^{i_2}) \ldots$. Note that $(\omega - F)(v_1, v_2, \ldots) = (\omega - F)(v_1^{j_1} e_{j_1}, \ldots) = v_1^{j_1} v_2^{j_2} \ldots (\omega - F)(e_{1,j_1}, e_{2,j_2}, \ldots) = 0$. $\square$

Let V_1, V_2 be vector spaces. We can bring multilinear maps into the framework of linear maps. Basically, we want to create a vector space $V_1 \otimes V_2$ formed by "formal" linear combinations of things of the type $v_1 \otimes v_2$.
Theorem: Suppose there exists a vector space (called the tensor product of V_i) $V_1 \otimes V_2$ and a multilinear map $\pi : V_1 \times V_2 \to V_1 \otimes V_2$ with the property that given any multilinear

map $T : V_1 \times V_2 \to W$, there is a unique *linear* map $\tilde{T} : V_1 \otimes V_2 \to W$ such that $T = \tilde{T} \circ \pi$. Then any other vector space satisfying this universal property is isomorphic to $V_1 \otimes V_2$ (with the isomorphism preserving the universal property).