MA 235 - Lecture 20

1 Recap

1. Differential of a function, pullbacks of 1-forms.
2. Tensors, dimension of tensors.

2 Tensors

Theorem: Suppose there exists a vector space (called the tensor product of V_i) $V_1 \otimes V_2$ and a multilinear map $\pi : V_1 \times V_2 \to V_1 \otimes V_2$ with the property that given any multilinear map $T : V_1 \times V_2 \to W$, there is a unique *linear* map $\tilde{T} : V_1 \otimes V_2 \to W$ such that $T = \tilde{T} \circ \pi$. Then any other vector space satisfying this universal property is isomorphic to $V_1 \otimes V_2$ (with the isomorphism preserving the universal property).
Proof: Suppose (V', π') is another such space. Then consider the map $\tilde{\pi'} : V_1 \otimes V_2 \to V'$ induced from π'. Likewise, we have $\tilde{\pi} : V' \to V_1 \otimes V_2$. These two are inverses of each other and hence give the desired isomorphism (why?). $\square$
We can prove that tensor products (if they exist) are associative (using the universal property). We can then take arbitrary (finite) number of tensor products. We need to manage to construct one such space. The idea is to take the free vector space $F(S = V_1 \times V_2 \times \ldots)$ defined as the set of all formal linear combinations of elements of S, i.e., $f : S \to \mathbb{R}$ such that $f(s) = 0$ for all but finitely many s. Define a subspace R generated by the set $(v_1, v_2, \ldots, av_i, \ldots) - a(v_1, v_2, \ldots)$ and $(v_1, v_2, \ldots, v_i + v_i', \ldots) - (v_1, \ldots, v_i, \ldots) - (v_1, \ldots, v_i', \ldots)$. The quotient space is denoted as $V_1 \otimes V_2 \ldots$ and the projection map by π. $\pi(v_1, v_2, \ldots)$ is denoted by $v_1 \otimes v_2 \ldots$. One can prove that indeed this satisfies the universal property.
One can also prove that if $e_{i,j}$ are bases for V_i, then $e_{1,j_1} \otimes e_{2,j_2} \ldots$ is a basis for the tensor product. Moreover, there is a canonical isomorphism between $V_1^* \otimes \ldots$ and $Mult(V_1, V_2, \ldots; \mathbb{R})$. Likewise (in finite-dimensions), there is a canonical isomorphism between $V_1 \otimes \ldots$ and $Mult(V_1^*, V_2^*, \ldots; \mathbb{R})$.
A covariant tensor of type-l on V is an element of $V^* \otimes V^* \otimes \ldots$ (l times). It can be thought of as corresponding to a multilinear map from $V \times V \ldots$ to $\mathbb{R}$. A contravariant tensor of type-k on V is an element of $V \otimes V \otimes \ldots$ (k times). A (k, l)-mixed tensor is an element of $V \otimes V \ldots (\ k\ times) \otimes V^* \ldots (\ l\ times)$. (By convention, $T^{0,0} = \mathbb{R}$.) In terms of indices, a (k, l) tensor has k upstairs indices and l downstairs indices.
Example: Given a f.d V, and $T : V \to V$, it can be thought of as a mixed $(1, 1)$-tensor, i.e., as an element of $V \otimes V^*$ as follows: Define $\mathcal{T} : V^* \times V \to \mathbb{R}$ as $\mathcal{T}(\omega, v) = \omega(T(v))$. This

is a multilinear map and hence corresponds to a unique linear functional on $V^* \otimes V$, i.e., to an element of $V \otimes V^*$. In fact, the map $T \to \mathcal{T}$ is a linear isomorphism from $L(V,V)$ to $V \otimes V^*$ (why?)
We will be interested in covariant tensors in this course. In fact, in elements of $T_p^*M \otimes T_p^*M \ldots$. An inner product on V is an example of a covariant 2-tensor, i.e., an element of $V^* \otimes V^*$. Indeed, it is a multilinear functional on $V \times V$. In fact, if $e_1, \ldots, e_n$ is an ordered basis of V and $e^1 \ldots$ is the dual basis, then $\langle , \rangle = \langle e_i, e_j \rangle e^i \otimes e^j = g_{ij} e^i \otimes e^j$. This example is very special. It is *symmetric*, i.e., $\langle v, w \rangle = \langle w, v \rangle$. On the other hand, suppose $v_1, \ldots, v_n$ are n elements of $\mathbb{R}^n$ forming the columns of a matrix A, then $\det(A)$ is a multilinear map from $\mathbb{R}^n \times \ldots$ to $\mathbb{R}$, i.e., a covariant tensor of type n. However, this one is antisymmetric/alternating, i.e., if you permute the elements, you pick up the sign of the permutation.

Def: A symmetric covariant tensor is one that is unchanged under a transposition of two of its entries (and hence under any permutation). An alternating/antisymmetric covariant tensor changes sign under a transposition (and hence picks up the sign of the permutation).
Example: A (skew)symmetric matrix gives a (skew)symmetric tensor: $A(v,w) = v^T A w$. Now $A = \frac{A+A^T}{2} + \frac{A-A^T}{2}$, i.e., every matrix is a sum of symmetric and antisymmetric matrices.
Motivated by this construction, define the symmetrisation of a k-covariant tensor α as $Sym(\alpha)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \alpha(v_{\sigma(1)}, \ldots)$. It is symmetric and $Sym(\alpha) = \alpha$ iff α is symmetric.
The antisymmetrisation/alternation is defined as $Alt(\alpha)((v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) \alpha(v_{\sigma(1)}, \ldots)$. It is alternating and $Alt(\alpha) = \alpha$ iff α is alternating.

3 Tensor bundles and tensor fields

Let $T_p^{k,l}M := T_pM \otimes T_pM \ldots T_pM \otimes T_p^*M \otimes T_p^*M \ldots$. The disjoint union $T^{k,l}M = \cup_{p \in M} T_p^{k,l}M$ can be given a vector bundle structure over M. This bundle is called the bundle of mixed (k,l)-tensors. Smooth sections of this bundle are called smooth (k,l)-tensor fields, i.e., smoothly varying tensor fields. Indeed, consider the obvious projection map to M. Each fibre is a vector space. Suppose (U,x) is a coordinate chart. Consider the basis $\frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes dx^{j_l}$. This basis gives a local trivialisation $\pi^{-1}(U) \to U \times \mathbb{R}^{n(k+l)}$. We declare the topology and manifold structure such that these local trivialisations are diffeomorphisms (as usual). As a consequence, a tensor field is smooth iff the coefficients in this trivialisation are smooth functions.

An example of a covariant symmetric 2-tensor field is a Riemannian metric: A Riemannian metric g on a smooth manifold M is a covariant symmetric 2-tensor field that defines an inner product on every tangent space.
Example: The Euclidean metric $g = (dx^1) \otimes (dx^1) + \ldots + (dx^n) \otimes (dx^n)$ on $\mathbb{R}^n$. A metric on $(0, \infty) \times (0, 2\pi)$: $g = dr \otimes dr + r^2 d\theta \otimes d\theta$. Note that this metric is basically the Euclidean metric on $\mathbb{R}^2$ but in different coordinates! This raises a question: Is every metric on $\mathbb{R}^n$ secretly the Euclidean metric locally in some coordinate chart? The answer is NO.

There is an obstruction called the Riemann curvature tensor.