MA 235 - Lecture 5

1 Recap

1. Implicit function theorem.
2. Lagrange's multipliers.
3. Topological manifolds (definition and a couple of examples).

2 Topological manifolds and smooth manifolds

Examples, non-examples:

- Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}$ be continuous. The *graph* M of f is $(x, f(x)) \subset \mathbb{R}^{n+1}$ with the subspace topology. Hence M is Hausdorff and second-countable. Consider $\phi(x, f(x)) = x$ to U. This map is a bijection. It is continuous. Its inverse is $\phi^{-1}(x) = (x, f(x))$ which is continuous. Hence M is a topological manifold that is homeomorphic to U. The chart ϕ is called "graph coordinates".
- Unfortunately, *closed* subsets of even $\mathbb{R}^n$ need not be topological manifolds: The letter X considered as a subspace of $\mathbb{R}^2$ cannot be endowed with a topological manifold structure (why?)

Recall that we want to do optimisation using calculus. To this end, we need to know what a differentiable function $f : M \to \mathbb{R}$ means. There is a naive way to define it. Simply use charts. f is diff iff $f \circ \phi^{-1}$ is so. Unfortunately, if we choose two different charts, then a function f can fail to be differentiable: Consider $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $(u, v) \to (u^{1/3}, v^{1/3})$ and $f(x, y) = x$. So change of charts must preserve differentiability or C^k-ness or smoothness or real-analyticity or anything else we feel like.

Definition: A smooth atlas for a topological manifold M is a collection of charts (ϕ, U_α) such that $\cup_\alpha U_\alpha = M$ and they are smoothly compatible with one another, i.e., if (ψ, U) and (ϕ, V) are charts, then $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is a smooth diffeomorphism. These maps are called *transition* maps.
Here is an odd thing for a smooth atlas: We would like to define a function $f : M \to \mathbb{R}$ to be smooth if $f \circ \phi^{-1}$ is smooth for every chart in the atlas. However, if we add *more* smoothly compatible charts, that does not change the smoothness or the lack thereof of functions! To remedy this odd point, we define a *maximal* atlas to be an atlas that is not properly contained in a larger smooth atlas.

Def: A *smooth structure* on a topological manifold M is simply a maximal smooth atlas. A smooth manifold $(M, \mathcal{A})$ is a topological manifold M equipped with a maximal smooth atlas $\mathcal{A}$.
As we shall see, a given topological manifold can have ostensibly different smooth structures (which are often "secretly" the same in disguise). Some topological manifolds (first found in 1960) can have *no* smooth structures at all!
Remark: One can also talk of C^k structures, real-analytic structures, complex structures, etc.

Theorem 1 (Forget about the adjective 'maximal' theorem)**.** *Let M be a topological manifold. Every smooth atlas $\mathcal{A}$ is contained in a* unique *maximal smooth atlas, called the smooth structure determined by $\mathcal{A}$. Two smooth atlases for M determine the same smooth structure iff their union is a smooth atlas.*

Proof. Let $\overline{\mathcal{A}}$ be the union of all charts that are *smoothly compatible* with $\mathcal{A}$. This beast is a smooth atlas compatible with $\mathcal{A}$: Indeed, if (ϕ, U), (ψ, V) are in $\overline{\mathcal{A}}$, then let $p \in U \cap V$. There is a chart $(\eta, W) \in \mathcal{A}$ that is smoothly compatible with (ϕ, U) and with (ψ, V). Thus $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ when restricted to $W \cap U \cap V$ is smooth. Thus (ϕ, U), (ψ, V) are smoothly compatible with each other.
Now $\overline{\mathcal{A}}$ is also maximal: Indeed, if there is any chart that is smoothly compatible with every element of $\overline{\mathcal{A}}$ such that it is *not* contained in $\overline{\mathcal{A}}$, then we have a contradiction.
If $\mathcal{B}$ is any other maximal smooth atlas containing $\mathcal{A}$, it is contained in $\overline{\mathcal{A}}$. Since $\mathcal{B}$ is maximal, $\mathcal{B} = \overline{\mathcal{A}}$.
If the union is a smooth atlas, then the unique maximal atlas containing the union contains each and hence we are done. If they determine the same smooth structure, then the maximal atlases are the same and hence their union is a smooth atlas. □

Just as in the case of topological manifolds, one talk of smooth coordinate balls, i.e., a member (ϕ, U) of the maximal smooth atlas, such that $\phi(U)$ is a ball in $\mathbb{R}^n$. (By the way, note that the closed coordinate balls are compact.)
Proposition: A smooth manifold M has a countable basis of smooth coordinate balls.
Proof: Firstly, every open cover V_α of M has a countable sub-cover: Consider a countable basis W_i. Let V_{α_j} be the countable subcollection such that V_{α_j} contains some basis element W_{i_j}. We claim that V_{α_j} cover M. Indeed, given any $p \in M$, $p \in V_\beta$ for some V_β. Since W_i form a basis, $p \in W_p \subset V_\beta$. Thus $V_\beta = V_{\alpha_i}$ for some i.
Secondly, using the above lemma, there is a countable cover of smoothly compatible coordinate charts (ϕ_i, U_i). Simply choose rational points and rational balls around them. □

Recall that when we use polar coordinates (r, θ), we simply write $p = (r, \theta)$ or $p = (x, y)$ depending on our convenience. (Technically, we are using two different $\mathbb{R}^2$s here.) Akin to that, in practice, one omits reference to ϕ, i.e., one says simply, "$p = (x^1, \ldots, x^n)$ in local coordinates". One *identifies* U with $\phi(U) \subset \mathbb{R}^n$ and abuses notation, i.e., one thinks of a manifold as a bunch of open subsets of (different) $\mathbb{R}^n$s with some "gluing" by transition functions (we will make this precise later on).
When we want to define objects on manifolds like say smooth functions to $\mathbb{R}$, we must make sure that our definitions are *independent* of choice of coordinates. There are two

ways of doing this: Either don't refer to coordinates at all while defining (the mathematician's way), or define using coordinates but make sure the "correct" results are obtained if we change coordinates (the physicist's way).

Examples of smooth manifolds:

- A countable discrete space (0 dimensional).
- $\mathbb{R}^n$ with the usual topology and usual chart.
- Open subsets of smooth manifolds. (As before, closed subsets need not be smooth (or for that matter, even topological) manifolds.
- Products of smooth manifolds $M_1 \times M_2 \ldots \times M_k$: Indeed, they are Hausdorff and second-countable. Suppose $(\phi_{\alpha,i}, U_{\alpha,i})$ are smooth atlases on M_i, then the "product chart" gives a smooth atlas on the product (that induces a unique smooth structure).
- Another smooth structure on $\mathbb{R}$: $U = \mathbb{R}$, $\phi : U \to \mathbb{R}$ is $\phi(u) = u^{1/3}$. This is a homeomorphism. Unfortunately, this chart is *not* smoothly compatible with the usual $\psi(u) = u$ because $\phi \circ \psi^{-1}(x) = x^{1/3}$ which is not smooth. So
- Finite-dimensonal normed vector spaces. (a choice of a basis identifies such an object with $\mathbb{R}^n$. A different basis gives a smoothly compatible chart.)
- Graphs of smooth functions $f : U \subset \mathbb{R}^n \to \mathbb{R}$.
- Matrices $M(m \times n, \mathbb{R}) = \mathbb{R}^{mn}$ and $M(m \times n, \mathbb{C}) = \mathbb{R}^{2mn}$.
- $GL(n, \mathbb{R})$: $\det(A) \neq 0$ is an open subset and hence a manifold. (Likewise, $GL(n, \mathbb{C})$ is also a manifold.)
- Matrices of full rank: Again, an open subset of $M(m \times n)$.