MA 235 - Lecture 6

1 Recap

1. Definition of smooth manifolds
2. "Forget the adjective maximal" theorem.
3. Examples.

2 Smooth manifolds

Examples (cont'd...)

1. Spheres S^n: Consider $\sum_i (y^i)^2 = 1$ as a subspace of $\mathbb{R}^{n+1}$. It is compact and Hausdorff. Here is a smooth atlas: Let U_i^+ be the open set where $y_i > 0$ and likewise for U_i^-. These sets cover the sphere. Now $U_i^\pm \cap S^n$ are graphs and hence possess graph coordinates: $\phi_i^\pm(y) = (y^1, y^2, \ldots, y^{i-1}, y^{i+1}, \ldots, y^n)$. hecking that transition maps are diffeomorphisms is an exercise.

2. Level sets: The example of spheres can be generalised. Let $U \subset \mathbb{R}^k$ be open and $f : U \to \mathbb{R}$ be smooth. Suppose $\nabla f(a) \neq 0$ whenever $f(a) = 0$. Then we claim that $f^{-1}(0)$ with the subspace topology can be made into a smooth manifold (HW).

3. Tori: $S^1 \times S^1 \ldots S^1$ is a torus.

4. Real projective spaces $\mathbb{RP}^n$: Consider the set of lines through the origin in $\mathbb{R}^{n+1}$. Every line can be given by a non-zero vector (upto scaling). Thus we have the quotient $\mathbb{RP}^n = \frac{\mathbb{R}^{n+1}-0}{X \sim \lambda X \mid \lambda \in \mathbb{R}_+}$. Endow this set with the quotient topology. It is Hausdorff (why?) Cover it with $U_i = \{[X] | X^i \neq 0\}$ (why are U_i open?). Consider $\phi_i : U_i \to \mathbb{R}^n$ given by $\phi_i([X]) = (\frac{X^0}{X^i}, \frac{X^1}{X^i}, \frac{X^{i-1}}{X^i}, \frac{X^{i+1}}{X^i}, \ldots)$. These ϕ_i are homeomorphisms. Hence $\mathbb{RP}^n$ is second-countable. The transition maps are smooth (why?) $\mathbb{RP}^n$ is also compact (why?)

3 Topological and smooth manifolds-with-boundary

We can have constrained optimisation problems where the the domain has a "boundary". We would want the boundary to also be "smooth". (So that we inductively apply our methods.) Just as manifolds are locally modelled on $\mathbb{R}^n$, near the boundary, manifolds-with-boundary must be locally modelled on the closed upper half-space $\mathbb{H}^n \subset \mathbb{R}^n$ ($x^n \geq 0$.) When $n > 0$, the topological boundary $\partial\mathbb{H}^n$ is $x^n = 0$, which is basically $\mathbb{R}^{n-1}$, i.e., we would want our boundaries to be manifolds themselves (without boundary). An n-dimensional topological manifold-with-boundary is a Hausdorff second-countable space that is either locally homeomorphic to an open subset of $\mathbb{R}^n$ (interior points and interior charts) or to a relatively open subset of $\mathbb{H}^n$ (boundary charts). The set of points sent to ∂H^n is called the boundary. It turns out (using invariance of domain) that $Int(M) \cap \partial M = \phi$ (why?).

A smooth atlas on a topological manifold-with-boundary is a collection of charts (ϕ_α, U_α) which are either interior or boundary charts such that the transition maps between any two are smooth. (A smooth function on an arbitrary set is one that can be extended to a smooth function on a neighbourhood of the set.) A smooth manifold-with-boundary is a topological manifold-with-boundary equipped with a maximal smooth atlas. Unfortunately, the product of two smooth manifolds-with-boundary is not considered a smooth manifold-with-boundary (it has corners). If there is a smooth chart $\phi : U \to \phi(U) \subset \mathbb{H}^n$ such that $\phi(p) \in \partial\mathbb{H}^n$ for some $p \in U$, then this is true for any chart containing p: If not, a neighbourhood of a boundary point of $\mathbb{H}^n$ can be diffeomorphed into an open subset of $\mathbb{R}^n$. This is a contradiction (why?)

4 Smooth functions on manifolds

Let M be a smooth manifold (with or without boundary). A function $f : M \to \mathbb{R}$ is said to be smooth at $p \in M$ if there *exists* a chart (ϕ, U) with $p \in U$ such that $\hat{f} = f \circ \phi^{-1} : \hat{U} \subset \mathbb{R}^n\{/\}\mathbb{H}^n \to \mathbb{R}$ is a smooth function at $\phi(p)$.
Immediately, we need to answer some obvious questions.
Proposition: Smoothness at p does not depend on the chart used and is a local property, i.e., suppose W is a neighbourhood of p; then f is smooth at p iff f restricted to W is smooth at p.
Proof: Firstly, locality holds for functions from open subsets of $\mathbb{R}^n\{/\}\mathbb{H}^n$ to $\mathbb{R}^m$ (as we shall see later). Now if (ψ, V) (with $p \in U \cap V$) is another chart, then on the open set $\psi(U \cap V)$, $f \circ \psi^{-1} = f \circ \phi^{-1} \circ (\phi \circ \psi^{-1})$ is smooth at $\psi(p)$ because it is a composition (and by locality). As for locality on manifolds, suppose f is smooth at p. Consider the chart $(\phi, W \cap U)$. $f \circ \phi^{-1} : \phi(W \cap U) \to \mathbb{R}$ is smooth. But that implies by definition that f restricted to W is smooth at p. The other direction is an exercise. $\square$
Proposition: Smooth functions are continuous.
Proof: $f \circ \phi^{-1} : \hat{U} \to \mathbb{R}$ is smooth and hence continuous at $\phi(p)$. Now $f = f \circ \phi^{-1} \circ \phi$ which is continuous at p.
A convention: Just as mentioned earlier, it is common practice to *identify* U with $\hat{U}$. Hence f with $\hat{f}$, i.e., the *local representation* of the function with the function itself. For instance, if $f(x, y) = x^2 + y^2$ on $\mathbb{R}^2$, one commonly writes $f(r, \theta) = r^2$ (whereas this is actually a different function).