MA 235 - Lecture 21

1 Recap

1. Tensor products of vector spaces and relationship with $Mult(V_1 \times V_2, \ldots, \mathbb{R})$.
2. Symmetric and skew-symmetric tensors. Symmetrisation and alternation.
3. Tensor bundles and tensor fields. Definition of a Riemannian metric.

2 Tensor bundles and tensor fields

Returning to alternating tensors, here is a useful result: α is alternating iff $\alpha(v_1, \ldots, v_k) = 0$ whenever the collection $v_1, \ldots, v_k$ is linearly dependent: Indeed, if the latter holds, in particular, if two of the v_i coincide, $\alpha(v_1, \ldots, v_k) = 0$. This means, $\alpha(v_1 + v_i, \ldots, v_i + v_1, \ldots,) = 0$. Thus, $\alpha(v_1, \ldots, v_i + v_1, \ldots) + \alpha(v_i, \ldots, v_i + v_1, \ldots) = 0$. Thus $\alpha(v_1, \ldots, v_i, \ldots) = -\alpha(v_i, \ldots, v_1, \ldots)$ and hence α is alternating.
Conversely, if α is alternating, and $\sum_i c_i v_i = 0$ with $c_1 \neq 0$ WLOG, then firstly, $\alpha(v_1, \ldots, v_k) = 0$ whenever two of the v_i coincide (why?) and hence $\alpha(c_1 v_1, \ldots, v_k) = \alpha(c_1 v_1 + c_2 v_2 + \ldots, v_2, \ldots) = 0$ (why?) Thus $\alpha(v_1, \ldots, v_k) = 0$ whenever they are linearly dependent. □

Let V be a f.d. vector space with a basis $e_1, \ldots, e_n$. Let $\epsilon^1, \ldots, \epsilon^n$ be the dual basis for V^*. Given a multi-index $I = (i_1, \ldots, i_k)$, consider the k-covariant tensor $\epsilon^I(v_1, \ldots, v_k) = \det \begin{pmatrix} \epsilon^{i_1}(v_1) & \ldots & \epsilon^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ \epsilon^{i_k}(v_1) & \ldots & \epsilon^{i_k}(v_k) \end{pmatrix} = \begin{pmatrix} v_1^{i_1} & \ldots & v_k^{i_1} \\ \vdots & \ddots & \vdots \\ v_1^{i_k} & \ldots & v_k^{i_k} \end{pmatrix}$, that is, $\epsilon^I(v_1, \ldots, v_k)$ is a minor of a matrix. These ϵ^I are called elementary alternating k-tensors or elementary k-forms. They are suppose to be (signed) volumes of some generalised parallelopipeds (as we shall see later on). For future use, if I, J are multiindices of size k, then we define δ^I_J as a determinant of a matrix $A_{ab} = (\delta)^{i_a}_{j_b}$. $\delta^I_J = sgn(\sigma)$ if $J = I_\sigma$ and no repetitions and 0 otherwise.
Properties:

- If I has a repeated index, $\epsilon^I = 0$ (why?).
- If $J = I_\sigma$, then $\epsilon^I = sgn(\sigma)\epsilon^J$ (why?).
- $\epsilon^I(e_{j_1}, \ldots, e_{j_k}) = \delta^I_J$ (why?).

Consider the vector space $\Lambda^k(V^*)$ - the space of alternating covariant k-tensors/k-forms on V. We want a basis of this space. Consider "increasing" multiindices, $i_1 < i_2 < \ldots$. For increasing-index-summation, we put a prime sign.
Theorem: Increasing-index elementary forms form a basis. As a consequence, $dim(\Lambda^k) = \binom{n}{k}$ when $k \leq n$ and 0 otherwise. Proof: If $k > n$, then by previous results, Λ^k is the trivial vector space (why?) So assume $k \leq n$. Firstly, the ϵ^I are linearly independent: If $\sum' c_I \epsilon^I = 0$, then consider $0 = \sum' c_I \epsilon^I(e_{j_1}, e_{j_2}, \ldots) = \sum' c_I \delta^I_J = c_J$ (why?) Secondly, they span the space: Let $\alpha \in \Lambda^k$. Then let $\alpha_I = \alpha(e_{i_1}, \ldots)$. Thus $(\alpha - \sum' \alpha_I \epsilon^I)(e_{j_1}, \ldots, e_{j_n}) = 0$ (why?) and hence we are done (why?). $\square$

If V is an n-dimensional v. space, then elements of $\Lambda^n(V^*)$ are called "top forms" (because there are no forms beyond them). Let $T : V \to V$ be a linear map and ω be a top form.
Then $\omega(Tv_1, \ldots, Tv_n) = \det(T)\omega(v_1, \ldots, v_n)$. Proof: Let $e_1, \ldots, e_n$ be a basis of V. We note that $\omega = c\epsilon^{12\ldots n}$ for some c. Since both sides are top forms, we only need to check when $v_i = e_i$. The RHS is $\det(T)c$. The LHS is $c\det(Te_1, \ldots, Te_n) = c\det(T)$ (why?). Hence we are done. $\square$

3 Wedge product

How does one generalise the cross product? Why must one generalise it?
Why: To talk perhaps of signed volumes in higher dimensions. Possibly to generalise the notion of curl $\nabla\times$ to formulate an FTC.
How: Naively, $(a \times b)_{ij} = a_i b_j - a_j b_i$, i.e., it is a 2-form! So perhaps we can talk of the "cross product" (we shall call it the wedge product) of a k-form with an l-form to get $\omega \wedge \eta$ - a $(k+l)$-form.

Def: Suppose $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$, then $\omega \wedge \eta := \frac{(k+l)!}{k!l!} Alt(\omega \otimes \eta)$.
So for instance, $\epsilon^1 \wedge \epsilon^2 = \epsilon^1 \otimes \epsilon^2 - \epsilon^2 \otimes \epsilon^1 = \epsilon^{1,2}$. So $\epsilon^1 \wedge \epsilon^2(v, w) = v^1 w^2 - v^2 w^1$.
Why the weird numerical factor? $(\epsilon^1 \wedge \epsilon^2) \wedge \epsilon^3 = \frac{3!}{2!1!} Alt((\epsilon^1 \otimes \epsilon^2 - \epsilon^2 \otimes \epsilon^1) \otimes \epsilon^3) = 3Alt(\epsilon^1 \otimes \epsilon^2 \otimes \epsilon^3 - \epsilon^2 \otimes \epsilon^1 \otimes \epsilon^3) = \sum_\sigma sgn(\sigma) \epsilon^{\sigma(1)} \otimes \epsilon^{\sigma(2)} \otimes \epsilon^{\sigma(3)} = \epsilon^{1,2,3} = \epsilon^1 \wedge (\epsilon^2 \wedge \epsilon^3)$.
Bear in mind that some old books don't have this factor. More generally,
Theorem: For any two multi-indices I, J, $\epsilon^I \wedge \epsilon^J = \epsilon^{IJ}$.
Proof: Let $P = (p_1, \ldots, p_{k+l})$. We need to show that $\epsilon^I \wedge \epsilon^J(e_{p_1}, \ldots) = \epsilon^{IJ}(e_{p_1}, \ldots)$ for all P. If P has repeated indices, by alternating-ness, both sides are zero. If P has an index that does not occur in I and J, then both sides are zero (why?) If P has no repeated indices, and $P = IJ$ (any permutation of it does not need to be checked), then $\epsilon^{IJ}(e_P) = 1$. For the LHS, $\epsilon^I \wedge \epsilon^J(e_P) = \frac{1}{k!l!} \sum_\sigma sgn(\sigma) \epsilon^I(e_{p_{\sigma(1)}}, \ldots, e_{p_{\sigma(k)}}) \epsilon^J(e_{p_{\sigma(k+1)}}, \ldots)$. The only surviving terms are of the type $\sigma = \tau\psi$. Thus $\epsilon^I \wedge \epsilon^J(e_P) = \frac{1}{k!l!} \sum_\tau sgn(\tau) \epsilon^I(e_{\tau(I)}) \sum_\psi sgn(\psi) \epsilon^J(e_{\psi(J)}) = 1$ (why?) $\square$