MA 235 - Lecture 22

1 Recap

1. Basis and dimension of $\Lambda^k(V^*)$.
2. How top forms change under linear maps.
3. Definition of the wedge product and proof that $\epsilon^I \wedge \epsilon^J = \epsilon^{IJ}$.

2 Wedge product

Properties of the wedge product:

- Bilinearity (Proof: Checking the definition).
- Associativity (Proof: Check for basis vectors).
- $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$ (Proof: Check for basis vectors).
- $\epsilon^{i_1} \wedge \epsilon^{i_2} \ldots = \epsilon^I$ (Proof: Inductively follows from associativity and the above theorem)
- $\omega^1 \wedge \omega^2 \ldots \omega^k(v_1, \ldots, v_k) = \det(\omega^i(v_j))$ (Proof: Induction and checking for the elementary ones).

It turns out that the wedge product is the unique such map satisfying the above properties. Caution: Not every form is a wedge of 1-forms (such forms are called decomposable). In $\mathbb{R}^3$ there is an identification of 2-forms with $\mathbb{R}^3$ itself and hence the cross product makes sense (but the choice of this identification matters. Sometimes $\vec{a} \times \vec{b}$ is called a pseudovector).

3 Differential forms on manifolds

We can take the disjoint union $\Lambda^k T^* M = \cup_{p \in M} \Lambda^k T_p^* M$. Suppose (U, x) is a chart. Since $\epsilon^i = dx^i$ is a basis for $T_p^* M$, whenever I is an increasing multi-index, $\epsilon^I = dx^{i_1} \wedge dx^{i_2} \ldots$ is a basis for $\Lambda^k T_p^* M$. We can give $\Lambda^k T^* M$ a vector bundle structure using these coordinate bases. A smooth section of this bundle of differential k-forms is called a k-form field (or simply a k-form). Such an object is a smooth linear combination of dx^I. We can define the wedge product of forms. Moreover, functions are treated as 0-forms.

$f \wedge \eta = f\eta$ if f is a function.

Suppose $F : M \to N$ is smooth. We can define the pullback as follows: If ω is a k-form field on N, $F^*\omega$ is a k-form field on M such that $(F^*\omega)_p(v_1, \ldots) = \omega_{F(p)}((F_*)_p(v_1), \ldots)$. For functions, by definition, $F^*f(p) = f(F(p)) = f \circ F(p)$. Recall that $F^*df = dF^*f$. Moreover, if $\omega = \omega_i dx^i$, then $F^*\omega = \omega_i \circ F dF^i$. For k-forms, the pullback is $\mathbb{R}$-linear (why?). $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$ (why?) Using this property, we can calculate pullbacks for several examples.
Suppose $\omega = f dy^1 \ldots dy^n$, then $F^*\omega = F^* f dF^1 \ldots dF^n$, which when acted on $\frac{\partial}{\partial x^1}, \ldots$ is $F \circ f \det(\frac{\partial F^i}{\partial x^j}) dx^1 \ldots dx^n$. In particular, $d\tilde{x}^1 \wedge \ldots = \det(\frac{\partial \tilde{x}^i}{\partial x^j}) dx^1 \wedge \ldots$.

4 The exterior derivative

How can we generalise the curl $\nabla \times$? Naively, we can try $d \wedge \omega$, i.e., pretend $d = \frac{\partial}{\partial x^i} dx^i$ is a "1-form" and take the "wedge product" with ω. This naive thing actually works!
Def: Let $\omega = \sum' \omega_I dx^I$ on $U \subset \mathbb{R}^n$. Then $d\omega := \sum' d\omega_I \wedge dx^I = \sum' \sum \frac{\partial \omega_I}{\partial x^k} dx^k \wedge dx^I$. This d is called the exterior derivative.
For 0-forms, i.e., functions f, df is the usual df defined earlier. For 1-forms ω, $d\omega = \sum_{i<j} (\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}) dx^i \wedge dx^j$. It coincides with the usual curl in $\mathbb{R}^3$. Properties:

- d is $\mathbb{R}$-linear. (Easy.)
- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$. ($d(\sum' \omega_I \eta_J dx^I \wedge dx^J) = \sum' \eta_J d\omega_I \wedge dx^I \wedge dx^J + \sum' \omega_I d\eta_J \wedge dx^I \wedge dx^J = d\omega \wedge \eta + \sum' (-1)^k \omega \wedge d\eta$.)
- $d^2 = d \circ d = 0$. (It is true for 0-forms (why?) So $d(d\omega) = d(d \sum' \omega_I \wedge dx^I) = 0 - d \sum' \omega_I d(dx^I) = 0$.)
- If $F : U \to V$ is a smooth map, then $F^*(d\omega) = d(F^*\omega)$. (So 0-forms, $F^*df(X) = dF^*f$ as before. Now $F^*(d \sum' \omega_I dx^I) = \sum' F^* d\omega_I \wedge F^* dx^I = \sum' dF^*\omega_I \wedge F^* dx^I = d(F^*\omega)$.)