MA 235 - Lecture 7

1 Recap

1. More examples of smooth manifolds (spheres, tori, real projective spaces)
2. Topological and smooth manifolds-with-boundary (no examples provided other than the trivial ones)
3. Definition of smooth functions $f : M \to \mathbb{R}$ and the locality property of smoothness.

2 Smooth functions on manifolds

Def: Let M, N be smooth manifolds or manifolds-with-boundary. $F : M \to N$ is said to be smooth at $p \in M$ if there exist charts (ϕ, U) (with $p \in U$) on M and (ψ, V) (with $F(p) \in V$) on N, such that $\psi \circ F \circ \phi^{-1} : \hat{U} \to \hat{V}$ is smooth at p. As before, we abuse notation often.
As before, if F is smooth, it is continuous. Smoothness is a local property. As a corollary, if U_α cover M and there are smooth maps $F_\alpha : U_\alpha \to N$ that agree on overlaps, then there is a unique smooth map $F : M \to N$ such that $F|_{U_\alpha} = F_\alpha$.
Properties:

- Constant maps $c : M \to N$ are smooth, $Id : M \to M$ is smooth, if $U \subset M$ is an open submanifold, then inclusion is smooth, compositions of smooth maps are smooth.
- Suppose $M_1, \ldots, M_k$ are smooth manifolds with or without boundary, such that at most one of them has a boundary. Then $M_1 \times \ldots M_k$ is a smooth manifold (possibly with boundary). For each i, let π_i be the projection. $F : N \to M_1 \times M_2 \ldots M_k$ is smooth iff $F_i = \pi_i \circ F$ are so (HW).

A digression: A useful smooth atlas for the spheres: Consider the open cover of S^n: $U^+ = \{x^{n+1} \neq 1\}$ and $U^- = \{x^{n+1} \neq -1\}$, i.e., the sphere minus the north pole, and minus the south pole. The stereographic projection: $\phi^+ : U^+ \to \mathbb{R}^n$ given by $\phi^+ = (\frac{x^1}{1-x^{n+1}}, \ldots, \frac{x^n}{1-x^{n+1}})$ and likewise $\phi^- = (\frac{x^1}{1+x^{n+1}}, \ldots, \frac{x^n}{1+x^{n+1}})$ are smoothly compatible charts. They are compatible with the usual charts too (exercise) and hence define the same smooth structure. In a sense, they are related to the notion that S^n is the one-point compactification of $\mathbb{R}^n$.
Examples of maps:

- Consider S^1 with the usual smooth structure. The function $f : \mathbb{R} \to S^1$ given by $f(t) = (\cos(t), \sin(t))$ is smooth. Indeed, consider the usual 4 graph charts on S^1. For instance, $y > 0$. Then $\phi(x, y) = x$ is the chart. So $f(t) = \cos(t)$ is the function which is smooth in this chart. Likewise for the others.

- $f : \mathbb{R}^n \to T^n$ given by $f(t) = (e^{it^1}, e^{it^2}, \ldots)$ is smooth.

- The inclusion map $i : S^n \to \mathbb{R}^{n+1}$ is smooth: Indeed, in the stereographic charts, we see that $\sum_i (y^i_+)^2 = \frac{1+x^{n+1}}{1-x^{n+1}}$ from which we can solve for x^{n+1} smoothly, and $i(y^1_+, \ldots) = ((1 - x^{n+1})y^1, (1 - x^{n+1})y^2, \ldots, (1 - x^{n+1})y^n, x^{n+1})$. Likewise for the other chart.

- The quotient map $\pi : \mathbb{R}^{n+1} - 0 \to \mathbb{RP}^n$ is smooth (why?)

- Define $q : S^n \to \mathbb{RP}^n$ by restriction of π. It is smooth (why?)

- What about a smooth map from S^n to T^n? (Hint: composition)

- From T^n to $\mathbb{R}^n$?

- From T^n to $\mathbb{S}^n$?

Diffeomorphisms: A smooth bijection $f : M \to N$ is called a diffeomorphism if f^{-1} is smooth. (The open interval is not diffeomorphic to a circle for instance.) The map $F : \mathbb{B}^n \to \mathbb{R}^n$ given by $F(x) = \frac{x}{1-|x|^2}$ is a diffeomorphism. Any smooth coordinate chart (ϕ, U) on M is actually a diffeomorphism between $U \subset M$ and $\phi(U)$ ($\subset \mathbb{H}^n$ or $\mathbb{R}^n$). Compositions of diffeos are diffeos, Cartesian products of diffeos are diffeos, diffeos are homeos, being diffeomorphic is an equivalence relation, the boundary is taken to the boundary under a diffeo. Recall the smooth structure $\phi(u) = u^3$ on $\mathbb{R}$? This structure is *not* the same as the usual one, but is *diffeomorphic* to it: $F : \mathbb{R} \to \tilde{\mathbb{R}}$ given by $F(x) = x^{1/3}$. In charts, it is $F(t) = t$ which is a diffeo.
Differential topology: The aim of differential topology is to classify (i.e., write a list) of "standard manifolds" with a way of telling whether a given manifold is diffeo to anything in the list. A 1-manifold is diffeo to either an open interval or a circle. If it has boundary, then to an interval or a half-line. A compact 2-manifold is diffeo to "a g-hold surface". Compact 3-manifolds are classified by geometrisation. For 4 and above, it is complicated.

3 Local to global - partitions of unity

Unfortunately, one cannot glue smooth functions that agree on closed subsets. On the other hand, it is helpful to construct lots of smooth functions. For instance, if one wants a bump function or perhaps a $1 - 1$ map from M to $\mathbb{R}^N$, and so on. More generally, one often has local functions f_α that one somehow wants to "blend together" to form a global one. To this end, it is helpful to have a partition-of-unity, i.e., a collection of smooth non-negative functions ϕ_α such that $\sum_\alpha \phi_\alpha = 1$ and there is a restriction on their supports.

Note that it makes sense to only sum up finitely many numbers. So it is helpful to have open covers U_α such that every point has a neighbourhood intersecting only finitely many sets. Such covers are called locally finite. Unfortunately, not every cover is locally finite (or even has a locally finite subcover): Consider $(-n, n)$ covering $\mathbb{R}$. The best we can do in this example is to take $(m, m+1)$, $(m-1/2, m+1/2)$. This cover is a *refinement* of the previous cover, i.e., every subset is in *some* U_α.
Paracompact space: Every open cover has a locally finite open refinement.
Proposition: Every smooth (in fact, just topological is enough) manifold is paracompact. (In fact, every metric space is so.) (Proof in the next class.)