MA 235 - Lecture 8

1 Recap

1. Smooth functions on manifolds, examples, diffeomorphisms.
2. Definition of paracompactness.

2 Partition-of-unity

Proposition: Every smooth (in fact, just topological is enough) manifold is paracompact. (In fact, every metric space is so.)
To prove the above proposition (and for other reasons), we need to "exhaust" the manifold by compact sets: A sequence K_i of compact sets is said to provide an exhaustion of a space X if $X = \cup_i K_i$ and $K_i \subset Int(K_{i+1})$.
Existence of an exhaustion of a topological manifold (with or without boundary): Consider the countable basis of coordinate balls or half-balls B_m (recall that their closures are compact). Let $K_1 = \bar{B}_1$. Suppose $K_1, \ldots, K_i$ have been found such that $K_j \subset Int(K_{j+1})$. Choose M_i so that $K_i \subset B_1 \cup B_2 \ldots B_{M_i}$. Assume that $M_i \geq i + 1$. Now $K_{i+1} := \bar{B}_1 \cup \bar{B}_2 \ldots$. Since $B_i \subset K_i$ (by inductive construction), we are done. $\square$

Returning to paracompactness, Proposition: Given a topological manifold M, an open cover χ of M, any basis $\mathcal{B}$ for M's topology, there exists a countable, locally finite refinement of χ, consisting of elements of $\mathcal{B}$. (Similar if boundary is there.)
Proof: Consider an exhaustion K_i. Let $V_j = K_{j+1} - Int(K_j)$. Cover the V_j's by finitely many elements of $\mathcal{B}$ such that each element is in $\mathcal{B}$ and in $W_j = K_{j+2} - Int(K_{j-1})$. Since $M = \cup_j V_j$, these elements cover M form a refinement, and since $W_j \cap W_k = \phi$ unless $j - 2 \leq k \leq j + 2$, it is locally finite. $\square$
Let $\chi = \{U_\alpha\}$ be an open cover of M. A partition-of-unity *subordinate* to χ is a family of smooth functions $\rho_\alpha : U_\alpha \to \mathbb{R}_{\geq 0}$ such that $0 \leq \rho_\alpha \leq 1$, $supp(\rho_\alpha) \subset U_\alpha$, the supports are locally finite, i.e. every point has a neighbourhood intersecting only finitely many of them, $\sum_\alpha \rho_\alpha = 1$.
Theorem: Suppose M is a smooth manifold with or without boundary. Let χ be an open cover of M. Then there exists a smooth partition of unity subordinate to it. There also exists a partition-of-unity consisting of compact supports subordinate to a locally finite countable open refinement.

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Theorem: Suppose M is a smooth manifold with or without boundary. Let χ be an open cover of M. Then there exists a smooth partition of unity subordinate to it. There also exists a partition-of-unity consisting of compact supports subordinate to a locally finite countable open refinement.
Proof: Assume that M does not have boundary. (The general case is similar.) Each U_α is a smooth manifold, and hence has a basis $\mathcal{B}_\alpha$ of coordinate balls (such that each ball is contained in a larger one). $\mathcal{B} = \cup_\alpha \mathcal{B}_\alpha$ is a basis for M. Thus there is a countable locally finite refinement B_i from $\mathcal{B}$. The closed cover $\bar{B}_i$ is also locally finite (why?). Consider a smooth bump function f_i that is > 0 on B_i and whose support is equal to $\bar{B}_i$. Define $f(x) = \sum_i f_i(x)$. The local finiteness of the cover implies that only finitely many f_i are non-zero near every point. Moreover, since B_i cover M, $f > 0$ on M. Define $g_i = \frac{f_i}{\sum_i f_i}$. Note that g_i form a partition-of-unity with compact supports subordinate to B_i. For every i, choose an index $a(i)$ so that $\bar{B}_i \subset U_{a(i)}$. For each α, define $\rho_\alpha = \sum_{a(i)=\alpha} g_i$. $supp(\rho_\alpha) = \cup_{i:a(i)=\alpha} \bar{B}_i \subset U_\alpha$. ☐
Applications:

- General bump functions: Let M be a smooth manifold (with or without boundary). For any closed $A \subset M$ and any open U (containing A), there is a smooth function $\psi : M \to [0, 1]$ such that $\psi \equiv 1$ on A and $supp(\psi) \subset U$.
  Proof: Consider the open cover of M given by U and A^c. Let $\rho_U, 1 - \rho_U$ be a partition-of-unity subordinate to this cover. Now $\psi = \rho_U$ is supported in U and equals 1 away from A^c, i.e., on A. ☐

- Extension of smooth functions: Let $A \subset M$ be closed and $f : A \to \mathbb{R}^k$ be smooth. Then for any open U containing A, there is a smooth $\tilde{f} : M \to \mathbb{R}^k$ such that $\tilde{f}|_A = f$ and $supp(\tilde{f}) \subset U$.
  Proof: For each $p \in A$, choose a neighbourhood $W_p \subset U$ and a smooth $\tilde{f}_p : W_p \to \mathbb{R}^k$ extending f from $A \cap W_p$ (by definition). The sets W_p, A^c form an open cover. Let ψ_p, ψ_0 be a smooth partition of unity. Define $\tilde{f} = \sum_p \psi_p \tilde{f}_p$. $\tilde{f}$ is the desired extension (why?). ☐