MA 235 - Lecture 23

1 Recap

1. Properties of the wedge product (including pullback)
2. Smooth form fields on manifolds.
3. Definition of the exterior derivative and properties:
   - d is $\mathbb{R}$-linear.
   - $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.
   - $d^2 = d \circ d = 0$.
   - If $F : U \to V$ is a smooth map, then $F^*(d\omega) = d(F^*\omega)$.

2 The exterior derivative

Suppose M is a smooth manifold with or without boundary.
Theorem: There are unique operators $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying the first three properties above and $df(X) = X(f)$. Moreover, in any chart, $d\omega = \sum' d\omega_I \wedge dx^I$.
Proof: If there are two such operators, $(d_1 - d_2)\omega = d_1\omega_I \wedge dx^I + 0 - d_2\omega_I \wedge dx^I + 0 = 0$. Define $d\omega := \phi^*(d_{\mathbb{R}^n}(\phi^{-1})^*\omega)$. $\mathbb{R}$-linearity is clear. $d(\omega \wedge \eta) = \phi^*(d_{\mathbb{R}^n}(\phi^{-1})^*(\omega \wedge \eta)) = \phi^*(d_{\mathbb{R}^n}(\phi^{-1})^*\omega \wedge (\phi^{-1})^*\eta) + (-1)^k\phi^*((\phi^{-1})^*\omega \wedge d_{\mathbb{R}^n}(\phi^{-1})^*\eta) = d\omega \wedge \eta + (-1)^k\omega \wedge d\eta$. As for the third property, $d \circ d\omega = \phi^*(d_{\mathbb{R}^n}(\phi^{-1})^*\phi^*(d_{\mathbb{R}^n}(\phi^{-1})^*\omega) = 0$. Lastly, $\phi^*(d_{\mathbb{R}^n}(\phi^{-1})^*f(X) = X(f)$ (why?) $d\omega$ is given by the expression above (why?) It is easy to show that $F^*d\omega = dF^*\omega$ (why?) $\square$

3 Closed forms and exact forms

In physics, a common question is if $\nabla \times \vec{F} = \vec{0}$, then is $\vec{F} = \nabla f$? The analogous question for forms is if $d\omega = 0$ (closed form), is $\omega = d\eta$ (exact form)?
Here is an example: $\omega = \frac{xdy - ydx}{x^2+y^2}$. $d\omega = 0$ (why?) but $\omega \neq df$. Indeed, if $\omega = df$, then $\frac{\partial f}{\partial x} = \frac{x}{x^2+y^2}, \frac{\partial f}{\partial y} = -\frac{y}{x^2+y^2}$. Consider $\int \nabla f . d\vec{r} = 0$ but it also equals $\int_0^{2\pi} d\theta = 2\pi$ (why?)
One can in fact prove that every closed 1-form on $\mathbb{R}^2 - 0$ is $c\omega + d\eta$ for some c. So it seems that this question has to do with the shape of the domain.
Poincaré lemma: Suppose ω is a closed k-form on $\mathbb{R}^n$, then it is exact.

De Rham cohomology: $H^k(M) = \frac{closed\ k-forms}{exact\ ones}$.
$H^0(M) = \mathbb{R}^k$ where k is the number of connected components (why?) $H^1(\mathbb{R}^2 - 0) = \mathbb{R}$. $H^k(M) = 0$ when $k > n$ (why?) $H^k(\mathbb{R}^n) = 0$ for $k > 0$. It turns out that the de Rham cohomology coincides with singular cohomology. So it is invariant under homeomorphism. (Thus showing how hard it is to distinguish between smooth structures.)

4 Integration in $\mathbb{R}^n$

There are two ways to integrate functions of more than one variable.
Riemann integral: One partitions a rectangle and defines the upper and lower Riemann sums of bounded functions as in the one variable case. One can prove that a function is Riemann integrable iff the set of discontinuities has measure zero. One can define the Riemann integral over arbitrary domains. Continuous bounded functions are Riemann integrable if the boundary has measure zero.
Lebesgue integral: One constructs the Lebesgue measure using volumes of rectangles. Then one integrates simple functions and approximates measurable functions by simple ones.
For functions with measure-zero discontinuities, these two coincide.

Fubini: Continuous functions on compact rectangles can be integrated one variable at a time in any order (Iterated integrals).
Fubini's theorem provides a way to actually compute integrals. Example: Integrate $f(x,y) = x^2 + y^2$ over $x^2 + y^2 \leq 1$. The circle has measure zero (why?) Extend f by 0 outside the circle. Fubini implies that $\int_{x^2+y^2\leq 1}(x^2+y^2)dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(x^2+y^2)dydx$ (why?) Thus it is $\int_{-1}^{1}(2x^2\sqrt{1-x^2} + \frac{2}{3}(1-x^2)^{3/2})dx$ which can be integrated (how?).

The volume of a parallelopiped P_n whose sides are $\vec{a}_i$ is $|\det(A)|$.
Sketch of proof: The Lebesgue measure is translation invariant. It is also invariant under orthonormal matrices (how?). Thus we can assume that $\vec{a}_1, \ldots, \vec{a}_{n-1}$ span the plane $x^n = 0$. $P_n = \{P_{n-1} + s\vec{a}_n \mid 0 \leq s \leq 1\}$. One can (exercise) "break" off a piece and translate to ensure that $vol(P_n) = vol(P_{n-1})a_n = |\det(A)|$. $\square$

Clearly the above integral could have been more easily evaluated in polar coordinates. But what does $dxdy$ change to? Morally, it ought to be $rdrd\theta$. We expect that a "small" rectangle with volume $dx^1dx^2\ldots$ under a change of variables $y(x)$ roughly changes to a small parallelopiped (when viewed in the x-coordinates) with edges $\frac{\partial \vec{y}}{\partial x^i}dx^i$. In the new coordinates, the volume is simply $dy^1dy^2\ldots$. Thus $dy^1dy^2\ldots = |\det\left(\frac{\partial y^i}{\partial x^j}\right)|dx^1dx^2\ldots$. In other words, we expect that $\int f(y)dV_y = \int f(y(x))|\det\left(\frac{\partial y^i}{\partial x^j}\right)|dV_x$.

Theorem: Let D, E be open bounded domains of integration in $\mathbb{R}^k$ (with boundaries of measure zero). Suppose $f : \bar{E} \to \mathbb{R}$ is a bounded continuous function. Let $G : \bar{D} \to \bar{E}$ be a smooth map that is a diffeo from D to E. Then $\int_E fdV = \int_D f \circ G|DG|dV$.
It turns out that (proof omitted) by an approximation argument, it is enough to consider

the case where f is a continuous compactly supported function on $\mathbb{R}^k$ such that its support lies in E. Before we embark on the proof, in one variable, the theorem would read as $\int_E f(y)dy = \int_D f(y(x))|y'(x)|dx$. The absolute value is puzzling! Let's look at an example: $\int_2^1 \cos(1/x)\frac{-dx}{x^2} = \int_{1/2}^1 \cos(y)dy = \sin(1) - \sin(1/2)$. The key point is that the limits are from 2 to 1! If we *insist* on limits being from lower numbers to higher numbers, *then* the integral is $\int_1^2 \cos(1/x)\frac{dx}{x^2} = \int_1^2 \cos(1/x)|\frac{-1}{x^2}|dx$.