MA 235 - Lecture 15

1 Level sets and Sard's theorem

Let M, N be smooth manifolds with or without boundary, $F : M \to N$ be a smooth map, and $S \subset M$ be an immersed or embedded submanifold, then $F : S \to N$ is also smooth. (Proof: Composition with inclusion.)
However, restricting to the codomain is more subtle. (Example: $G(t) = (\sin(2t), \sin(t))$ with its domain extended to $\mathbb{R}$ is not continuous to the figure-8 but is smooth when treated as a map to $\mathbb{R}^2$.) Moreover, if the codomain has a boundary, again it is a tricky affair.
This is not a problem for *embedded* submanifolds (without boundary): Let $S \subset M$ be an embedded submanifold and M be a manifold. Let N be a manifold. Then if $F : N \to M$ is a smooth map such that $F(N) \subset S$, then $F : N \to S$ is a smooth map. (HW)
Using these results we can show that submanifolds have a unique smooth structure.

Recall that S^n was defined as $\sum(x^i)^2 = 1$. Does this mean that if we set our favourite smooth function to 0, we will get a compact $n-1$-dimensional submanifold of $\mathbb{R}^n$? Nope. There are several kinds of counterexamples:

1. It need not be compact: Take $x = 0$.

2. It can be empty: $x^2 + y^2 + 1 = 0$. (By the way, empty sets are manifolds of any dimension by definition!)

3. It need not even be a topological manifold: $x^2 - y^2 = 0$.

4. It need not be a submanifold: $y^2 - x^3 = 0$. Indeed, if this set were a submanifold near the origin, then near the origin, we can change coordinates to (u, v) so that $v = 0$ is this subset,i .e., this subset is $u \to (x(u, 0), y(u, 0))$. Suppose $\frac{\partial x}{\partial u} \neq 0$ at the origin. Then changing charts to (x, v), we see that $y = y(u, v) = y(u(x, v), v)$ and hence $y^2 = x^3$ near the origin iff $y = y(u(x, 0), 0)$, i.e., y is a smooth function of x. But that is impossible. (Likewise, if $\frac{\partial x}{\partial u} = 0$ at the origin, then x is a smooth function of y.)

Compactness and emptyness aside, the main problem appears to be that $\nabla f = 0$ at some points where $f = 0$. (Caution: This is not a *necessary* condition by any means! Take $x^2 = 0$. It is a submanifold!)
Def: Let M, N be smooth manifolds (without boundary) and $F : M \to N$ be a smooth map. A point $p \in M$ is a *regular point* of F if $(F_*)_p$ is surjective. Otherwise, it is a *critical*

point of F. A regular *value* of F is a point $c \in N$ such that *every* point in $F^{-1}(c) \subset M$ is a regular point of F. A critical *value* of F is a point $c \in N$ such that it is not a regular value, i.e., $F^{-1}(c)$ has at least one critical point. If c is a regular value, then $F^{-1}(c)$ is a regular level set. Note that if $F^{-1}(c) = \emptyset$, then c is a regular value.
Theorem: Every regular level set of a smooth map between smooth manifolds is an embedded submanifold whose codimension equals the codimension of the codomain.
Proof: Let $S = F^{-1}(c) \subset M$. For every $p \in S$, choose arbitrary charts centred $p, F(p) = c$. Thus $[DF]$ has constant rank. Hence we can change charts (to centred ones) so that $F(x) = (x^1, \ldots, x^n)$. Thus S is an $n-m$-slice near p. Thus S is an embedded submanifold with dimension $n-m$. $\square$
If in addition, F is a *proper map*, that is $F^{-1}(compact) = compact$, then if c is a regular value, $F^{-1}(c)$ is compact submanifold. In fact, a regular level set is also a properly embedded submanifold, i.e., the inclusion map is proper. Indeed, $F^{-1}(c)$ is a closed subset by continuity. If $K \subset M$ is compact, then $K \cap F^{-1}(c)$ is compact. Hence $i^{-1}(F^{-1}(c))$ is compact.

$S \subset M$ is a submanifold of dimension k iff locally it is the level set of a submersion $F : U \to \mathbb{R}^{m-k}$. (Such a function is called a local defining function.)
Proof: If S is a submanifold: There are local slice charts. Take $F = (x^{k+1}, \ldots, x^n)$ for such charts.
If there is a local defining function: It is locally a submanifold and hence satisfies the local slice condition. Thus it is a submanifold. $\square$
It is not true that the codimension-1 submanifold of $\mathbb{R}^n$ has a global defining function. However, under some necessary condition (nowhere vanishing smoothly varying unit normal), it is true. Under a similar necessary condition, it is harder to prove but true that a codimension-2 submanifold of $\mathbb{R}^n$ has a defining function. As far as I know, this problem is open for higher codimensions.

Do regular values exist at all?
Sard's theorem (a weak version): For a smooth map $F : M \to N$, the set of regular values is *dense* in N.
In particular, if $f : M \to \mathbb{R}$ is a smooth exhaustion, then there is an increasing sequence $c_i \to \infty$ such that $f^{-1}(c_i)$ is a smooth manifold and $f^{-1}(-\infty, c_i]$ form an exhaustion. In fact, $f^{-1}(-\infty, c_i]$ form manifolds-with-boundary (why?). There cannot be an onto smooth map from $\mathbb{R}$ to $\mathbb{R}^2$: Indeed, if there is such a map, then there is a $c \in \mathbb{R}^2$ such that $f^{-1}(c)$ is regular level set. Hence $f^{-1}(c)$ is a submanifold of dimension $1-2=-1$! A contradiction. On the other hand, there are continuous space filling curves.

2 Vector fields and the tangent bundle

Recall that one of the points of studying tangent vectors was to model particles flowing on a manifold. For instance, suppose we take the flow of a fluid (maybe electrons or even water) on a surface (like say the surface of a metallic ball). The velocities of the particles vary smoothly. Moreover, suppose we consider the flow for one second. Then a particle at p goes to some other point. This operation is reversible. We can hope that

it is a diffeomorphism of the manifold! Lastly, suppose we cover the sphere S^2 with hair. Can we comb the sphere so that no hair sticks out completely? The answer is no (the hairy ball theorem).
All of the above need a notion of smoothly varying tangent vectors (such an object is called a smooth vector field).

If a manifold M is a submanifold of $\mathbb{R}^N$, then we can easily define a notion of smoothly varying tangent vectors: Simply take a smooth function $X : M \to \mathbb{R}^N$ such that $X(p)$ lies in the tangent plane at p. On paper, using Whitney embedding, we can make this definition on any manifold. But we want a more intrinsic definition (without reference to a particular embedding).

Def: Let $TM = \cup_{p \in M} T_p M$. A function $X : M \to TM$ such that $X(p) \in T_p M$ is called a vector field.

We want to define a smooth vector field. At least locally, can we come up with a reasonable *example* of a smooth vector field? A natural choice is the coordinate basis $\frac{\partial}{\partial x^i}$.

Def: Let M be a smooth manifold (with or without boundary). A vector field $X : M \to TM$ is smooth at p if there exists a coordinate chart (U, x) near p such that $X = X^i \frac{\partial}{\partial x^i}$ where the functions $X^i : U \to \mathbb{R}$ are smooth at p. A smooth vector field is one that is smooth at all points.

The definition of smoothness is independent of the choice of coordinate chart: Suppose $(\tilde{U}, \tilde{x})$ is another coordinate chart around p, then on $U \cap \tilde{U}$, $\tilde{X}^i = \frac{\partial \tilde{x}^i}{\partial x^j} X^j$. Since the coefficients are smooth, $\tilde{X}^i$ is a linear combination of functions that are smooth at p, and hence $\tilde{X}^i$ are smooth at p.