MA 235 - Lecture 24

1 Recap

1. Exterior derivative on manifolds-with-boundary.
2. Riemann and Lebesgue integrals. Fubini's theorem. Example.
3. Statement of change of variables formula. "Paradox" with one variable substitution.

2 Integration in $\mathbb{R}^n$

Theorem: Let D, E be open bounded domains of integration in $\mathbb{R}^k$ (with boundaries of measure zero). Suppose $f : \bar{E} \to \mathbb{R}$ is a bounded continuous function. Let $G : \bar{D} \to \bar{E}$ be a smooth map that is a diffeo from D to E. Then $\int_E f dV = \int_D f \circ G |DG| dV$.
It turns out that (proof omitted) by an approximation argument, it is enough to consider the case where f is a continuous compactly supported function on $\mathbb{R}^k$ such that its support lies in E. Before we embark on the proof, in one variable, the theorem would read as $\int_E f(y)dy = \int_D f(y(x))|y'(x)|dx$. The absolute value is puzzling! Let's look at an example: $\int_2^1 \cos(1/x)\frac{-dx}{x^2} = \int_{1/2}^1 \cos(y)dy = \sin(1) - \sin(1/2)$. The key point is that the limits are from 2 to 1! If we *insist* on limits being from lower numbers to higher numbers, *then* the integral is $\int_1^2 \cos(1/x)\frac{dx}{x^2} = \int_1^2 \cos(1/x)|\frac{-1}{x^2}|dx$.

However, if we want to take $dy = y'(x)dx$ more seriously, then dy, dx ought to be 1-forms. That being said, if we want to define integration only for 1-forms (as opposed to functions), then we must restrict ourselves to change of variables whose y' is > 0. We shall return to this connection with forms later.

Example: Evaluate $\int_{B(0,1)} x^2y^2 dA$ if it is Lebesgue integrable.
Note that the boundary of the disc has measure zero (why?) and x^2y^2 is continuous and bounded on the disc. Hence it is Riemann integrable and thus Lebesgue integrable. Now if we throw out the x-axis (and the origin), the integral does not change because that set has measure zero. Now $G : (0,1) \times (0, 2\pi) \to B(0,1) - \{x - axis\}$ given by $G(r,\theta) = (r\cos(\theta), r\sin(\theta))$ is smooth, 1-1, onto (why?) and $\det(DG) = r > 0$. Thus IFT implies that the inverse is smooth. Moreover, $G : [0,1] \times [0, 2\pi] \to B(\bar{0}, 1)$ is smooth and the boundary has measure zero. Thus by the change of variables formula, the given integral is $\int_{(0,1)\times(0,2\pi)} r^5 \cos^2\theta \sin^2\theta$ which can be evaluated using the Fubini theorem

and the one-variable FTC.

A sketch of proof of change of variables: We want to first prove the result for some special kinds of change of variables. Then we want to write *any* G as a composition of these special kinds. Firstly, by Fubini, the theorem is true when G merely exchanges two coordinates. Secondly, this theorem is true if we consider *primitive* mappings, that is, ones that change at most one coordinate, i.e., $G(x) = (x^1, \ldots, x^{i-1}, g(x), x^{i+1}, \ldots)$ (why?). Thirdly, if the theorem is true for P, Q, then if $S = P \circ Q$, $\int f(z)dz = \int f(P(y))|DP(y)|dy = \int f(S(x))|DP(Q(x))||DQ(x)|dx = \int f(S(x))|DS(x)|dx$. Fourthly, it turns out (proof in Rudin by IFT one variable at a time) that every T is *locally* $T(x) = T(a) + B_1 \circ \ldots B_{k-1} G_{k-1} \circ \ldots G_1(x-a)$ where B_i are identity or flips, and G_i are primitive. So if the support of f lies in a small neighbourhood where this decomposition holds, we are done. Otherwise use a partition-of-unity.

3 Orientability

Ideally, on a manifold, we would like to define the integral of a function $f : M \to \mathbb{R}$ as $\int f = \sum_i \int \rho_i f dx^1 dx^2 \ldots$ where ρ_i are partitions-of-unity. The problem is that if we change coordinates, then the integrals change! Heck even in $\mathbb{R}^n$, if f is compactly supported, if we take a smooth diffeo $T : \mathbb{R}^n \to \mathbb{R}^n$, then the integrals will not be the same. The modulus of the Jacobian kicks in. So the bottom line is that we cannot hope to define the integral of a *function* $f : M \to \mathbb{R}$. However, taking the dx^i seriously as 1-forms, we notice that the Jacobian factor is almost exactly how forms change under coordinate changes!
Thus, to begin with, let U be an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$. Let $\omega = f dx^1 \wedge dx^2 \wedge \ldots$ be an n-form that is compactly supported on U. We define $\int_U \omega := \int_U f dx^1 dx^2 \ldots$ (simply erase the wedges!).

On the other hand, in $\mathbb{R}^n$ we can define the integrals of top forms. So we could try $\int_M \omega = \sum_i \int_{\mathbb{R}^n} \rho_i f dx^1 dx^2 \ldots$. The only problem is that the sign of the Jacobian plays a role in the change of variables formula. What if we could cover M by coordinate charts such that the Jacobians are all positive? In this case, we have some hope. "Def" (Warning: This definition is useful when $dim(M) > 1$ or $\partial M = \phi$.): Suppose M is a smooth manifold (with or without boundary) and (x_α, U_α) is a smooth atlas consisting of connected charts such that $\det(\frac{\partial x^i_\alpha}{\partial x^j_\beta}) > 0 \; on \; U_\alpha \cap U_\beta$ for all α, β, then we say that M is equipped with an oriented atlas/ M has a given orientation. (If such an atlas exists, then we say that M is orientable.)

When do we say that two such atlases give the "same" orientation?
Def: Two smooth oriented atlases $\mathcal{A}$ and $\mathcal{B}$ are said to be compatible orientation-wise/ define the same orientation if $\mathcal{A} \cup \mathcal{B}$ is an oriented atlas.
Suppose M is orientable. Then orientation-compatibility is an equivalence relation among oriented atlases (why?) To determine the number of equivalence classes, we need a more concise interpretation of orientation.
Given an oriented manifold $(M, (x_\alpha, U_\alpha))$, let ρ_α be a partition-of-unity subordinate to

the atlas. Define $\omega = \sum_\alpha \rho_\alpha dx_\alpha^1 \wedge dx_\alpha^2 \ldots$. Note that $\omega \neq 0$ anywhere (why?) Moreover, ω is a positive multiple of $dx_\alpha^1 \wedge dx_\alpha^2 \ldots$ for all α. Conversely, suppose M either does not have a boundary or $dim(M) > 1$. Also suppose ω is a nowhere vanishing top form, and suppose $\mathcal{A}$ is any atlas consisting of connected charts. We can change the charts and produce a new atlas to make sure that $\omega(\frac{\partial}{\partial x_\alpha^1}, \frac{\partial}{\partial x_\alpha^2}, \ldots) > 0$ for all α. Indeed, if $\dim(M) > 1$, then simply interchange two coordinates. That flips the sign. (Of course, prove that this "fix" actually produces a collection of oriented charts.) If $\dim(M) = 1$ and $\partial M = \phi$, then take $t_\alpha \to -t_\alpha$. (This does not work if one has a boundary. Why?)
So a manifold (such that $dim(M) > 1$ or $\partial M = \phi$) is orientable iff it admits a nowhere vanishing top form. We say that a chart is compatible with an orientation form ω if $\omega(\frac{\partial}{\partial x^1}, \ldots) > 0$ at all points.

In fact, define an equivalence relation between nowhere vanishing top forms: $\omega \sim \omega'$ if $\omega = f\omega'$ where $f > 0$. Then if M is connected we have exactly two equivalence classes (why?)
The above correspondence gives a bijection between the two sets of equivalence classes when M has no boundary or when $dim(M) > 1$, i.e., Given $[(x_\alpha, U_\alpha)]$ consider $[\sum_\alpha \rho_\alpha dx_\alpha^1 \wedge \ldots]$. Firstly, this map is well-defined. Secondly, it is onto (why?) Thirdly, it is $1-1$: If $\frac{\sum_\alpha \rho_\alpha dx_\alpha^1 \wedge \ldots}{\sum_{\alpha'} \rho'_\alpha dy_{\alpha'}^1 \wedge \ldots} > 0$, and if these two atlases are not compatible then $\det(\frac{\partial x_\alpha^i}{\partial y_{\beta'}^j}) < 0$ for some α, β' throughout $U_\alpha \cap U_{\beta'}$ (why?). This means that the above ratio must be negative in this region (why?) Thus we have a contradiction. $\square$

The case when $dim(M) = 1$ and $\partial M \neq \phi$: The above correspondence means that we have exactly two equivalence classes for orientation when $\partial M = \phi$ or $dim(M) > 1$. Often, one arbitrarily designates one class as "positively oriented" and the other as negatively oriented. Unfortunately, in this case since we have defined the boundary chart to have positive last coordinate, our definition of orientation is not a nice one. To avoid this problem, one *defines* orientation of manifolds using the existence of nowhere vanishing top forms. Then every orientable manifold (with or without boundary) has exactly two orientation classes. When $\partial M = \phi$ or $dim(M) > 1$, this corresponds to orienting using coordinate charts.