# MA 235 - Lecture 25

## 1 Recap

1. Sketch of proof of change of variables.
2. Example.
3. Definition of orientability using charts for $dim(M) > 1$ or $\partial M = \phi$.
4. Equivalence with the presence of nowhere vanishing top-forms.
5. Equivalence relations of orientations and presence of only two equivalence classes on connected manifolds.
6. Definition using nowhere vanishing top-forms for $\dim(M) = 1$ and $\partial M \neq \phi$.

## 2 Orientability

Examples of orientable manifolds:

- $\mathbb{R}^n$ is orientable.
- A codimension-0 submanifold $D \subset M$ is orientable if $M$ is so: Suppose $\omega$ is an orientation form on $M$, then $i^*\omega$ is one on $D$.
- If $M, N$ are orientable, then so is $M \times N$ with the "product orientation": Take $\pi_1^*\omega_1 \wedge \pi_2^*\omega_2$ as the orientation form.
- Suppose $F : M \to N$ (where $M, N$ are connected with $dim > 0$) is a smooth map such that $F_*$ is invertible at all points. If $(F_*)_p$ is orientation-preserving at all points, then $F$ is said to be orientation-preserving. Otherwise it is said to be orientation-reversing. Given an orientation $[\omega]$ on $N$, there is a unique orientation ( called the pullback orientation) such that $F$ is orientation-preserving: $[F^*\omega]$ does the job (why?). If $[\eta]$ is any other such orientation, then $\omega(F_*e_1, F_*e_2, \ldots)/\eta(e_1, \ldots) > 0$ (why?). Thus $[\eta] = [F^*\omega]$.
- Hypersurfaces in $M$: Suppose $(M, [\omega])$ is an oriented smooth manifold with or without boundary, and $S \subset M$ is a smooth hypersurface (without boundary that does not intersect $\partial M$). Suppose $\vec{N}$ is a section of $TM$ restricted to $S$ such that $\vec{N}$ is nowhere tangent to $S$. Then $S$ is orientable with the orientation given by

the form $(e_1, \ldots, e_{n-1}) \to \omega(\vec{N}, e_1, \ldots)$ ( Indeed, $\vec{N}, e_1 \ldots$ are linearly independent and hence $\omega(\vec{N}, \ldots) \neq 0$.) For instance, $S^n$ can be oriented this way.

Induced orientation on $\partial M$: Proposition: Let $M$ be an oriented smooth $n$-fold with boundary ($n \geq 1$). Then $\partial M$ is orientable, and all "outward pointing" vector fields ( Smooth sections $X$ of $TM$-restricted-to-$\partial M$ such that $X = c_n \frac{\partial}{\partial x^n} + \ldots$ where $c_n < 0$ whenever $(x^1, \ldots)$ is a boundary chart) along $\partial M$ determine the same orientation.
Proof: Firstly, $\partial M$ is a smooth hypersurface: Cover it with boundary charts for $M$. Now the first $n-1$ coordinates provide charts for $\partial M$ (why?). Secondly, there is a smooth outward pointing vector field on $\partial M$: Take a partition-of-unity and define $X = \sum_\alpha -\rho_\alpha \frac{\partial}{\partial x^n_\alpha}$. Thirdly, use the induced orientation $\omega(X_{outward}, \ldots)$. Fourthly, suppose $X_1, X_2$ are two such outward fields, then $\omega(X_1, \frac{\partial}{\partial x^1}, \ldots, ) = c_n \omega(\frac{\partial}{\partial x^n}, \ldots)$ which is a positive multiple of $\omega(X_2, \ldots)$ (why?) $\square$
As an example, the orientation on $\mathbb{H}^n$ is the same orientation as that of $\mathbb{R}^{n-1}$ only when $n$ is even (why?).

Non-examples:

- It turns out that $\mathbb{RP}^n$ is orientable iff $n$ is odd. For instance, $\mathbb{RP}^2$ is not orientable.
- The Möbius line bundle is not orientable.

# 3 Orientation and integration of top forms

Proposition: Suppose $D, E$ are domains of integration in $\mathbb{R}^n$ or $\mathbb{H}^n$ and $G : \bar{D} \to \bar{E}$ is a smooth map that is a diffeo from $D$ to $E$. If $\omega$ is a smooth top form on $\bar{E}$, then $\int_D G^*\omega = \int_E \omega$ if $G$ is orientation-preserving and $-\int_E \omega$ if $G$ otherwise.
Proof: Follows from the change of variables formula (how?) $\square$

Proposition: Suppose $U, V$ are open subsets in $\mathbb{R}^n$ or $\mathbb{H}^n$ and $G : U \to V$ is a diffeo. If $\omega$ is a smooth top form with compact support in $V$, then $\int_U G^*\omega = \int_V \omega$ if $G$ is orientation-preserving and $-\int_U \omega$ if $G$ otherwise.
Proof: Let $E \subset \bar{E} \subset V$ be a domain of integration containing the support of $\omega$. (Why does it exist?) Now $G^{-1}(E)$ is a domain of integration too (why?) Hence we are done (why?) $\square$