MA 235 - Lecture 9

1 Recap

1. Partitions-of-unity.

2. Applications - Existence of bump functions (Urysohn type) and local-smooth-extensions (from closed subsets) implies global-smooth-extension.

2 Applications of partitions-of-unity (cont'd...)

- Existence of smooth exhaustion functions: Every smooth manifold (with or without boundary) admits a smooth positive exhaustion function, i.e., a smooth function $f : M \to \mathbb{R}$ such that $f > 0$, $f^{-1}((-\infty, c])$ is compact for all $c \in \mathbb{R}$. (The sets $f^{-1}((-\infty, n])$ form an exhaustion.)
  Proof: Let V_j be any countable pre-compact open cover. Let ψ_j be a smooth partition of unity subordinate to V_j. Define $f = \sum_j j\psi_j$. This function is smooth and positive (why?) If $c \in \mathbb{R}$, choose an integer $N > c$. If $p \notin \cup_{j=1}^{N} \bar{V}_j$, then $\psi_j(p) = 0$ for all $j \leq N$. Thus $f(p) > c$ (why?). We are done (why?) $\square$

- Level sets of smooth functions (proof omitted): Let M be a smooth manifold. If $K \subset M$ is closed, there is a smooth $f : M \to [0, \infty)$ such that $f^{-1}(0) = K$. $\square$

3 Tangent vectors and tangent spaces

Recall that we want to optimise smooth functions over manifolds. Naively, we might expect some sort of Lagrange's multipliers theorem but for that one might need to make sense of vectors "tangent" to the manifold. (Recall that the gradient gives us the normal to a regular level set.) Another reason to study tangent vectors is that suppose we want to look at the motion of a ring on a wire or electrons on a two-dim surface for instance, then their velocities are constrained to be "tangent" to the constraining surfaces. What is a vector "tangent" to a sphere S^n at $p \in S^n$? Presumably it is the velocity of a particle moving on it. In other words, a tangent vector lies on a tangent plane but the plane keeps moving from point to point. So we have several "tangent spaces" that vary from point to point. Unfortunately, a general manifold is not defined as "sitting inside" $\mathbb{R}^N$ like S^n is. So how can we define "tangent vectors"? There is a way to do it using velocities of curves, but we shall come to it later.

3.1 Tangent vectors through functions

The only way to "probe" a manifold is by means of smooth functions. The point of the tangent plane/tangent vectors is to provide a linear approximation to the manifold. Likewise, can we hope that tangent vectors can be deduced by knowing linear approximations of smooth functions? For instance, in $\mathbb{R}^n$, the linear approximation of a smooth function can be deduced if we know all directional derivatives. The directional derivative $D_{a,v}f = \frac{\partial f}{\partial x^i}(a)v^i$. So we can "read off" the components of tangent vectors from directional derivatives of smooth functions. So what properties characterise directional derivatives?

- A directional derivative $D_{a,v}$ takes smooth functions on $\mathbb{R}^n$ to numbers in a linear manner.
- But the crucial point is that functions can be multiplied. $D_{a,v}(fg) = f(a)D_{a,v}g + D_{a,v}fg(a)$.
- Are these properties enough?

Def: A derivation D at $a \in \mathbb{R}^n$ is a linear map over $\mathbb{R}$ $D : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathbb{R}$ such that $D(fg) = f(a)Dg + Dfg(a)$.
$D_{a,v}$ is an example of a derivation.
Proposition: Derivations form a vector space $T_a\mathbb{R}^n$, every derivation is of the form $D(f) = D_{a,v}f$ for some v, and $v \to D_{a,v}$ is a linear isomorphism between $\mathbb{R}^n$ and $T_a\mathbb{R}^n$.
Proof: Define the vector space structure as $(\alpha D_1 + \beta D_2)f = \alpha D_1 f + \beta D_2 f$. Given D, define $v^i = D(x^i)$. Consider the derivation $w = D - D_{a,v}$. $w(x^i) = 0$. Moreover, $D(1.1) = 2.D(1)$ and hence $D(1) = 0$. If c is a constant, $D(c) = cD(1) = 0$. Moreover, $f = f(a) + \frac{\partial f}{\partial x^i}(a)(x^i - a^i) + h_{i,j}(x,a)(x^i - a^i)(x^j - a^j)$ for some smooth $h_{i,j}$. Thus, $w(f) = w(h_{i,j}(x,a)(x^i - a^i)(x^j - a^j))$ which equals 0 (why?) Thus, $D = D_{a,v}$. The map $v \to D_{a,v}$ is clearly linear (why?) and onto. Moreover, if $D_{a,v}f = 0$ for all smooth f, then $v = D_{a,v}(x^i)e_i = 0$. Thus it is a linear isomorphism.
Corollary: The derivations $\frac{\partial}{\partial x^i}|_a$ defined by $\frac{\partial}{\partial x^i}|_a f = \frac{\partial f}{\partial x^i}(a)$ form a basis for $T_a\mathbb{R}^n$.

4 Tangent vectors on manifolds and pushforwards

Let M be smooth manifold (with or without boundary). A linear map $w : \mathcal{C}^\infty(M) \to \mathbb{R}$ is called a derivation at p, if $w(fg) = w(f)g(p) + f(p)w(g)$. The set of all derivations at p can be made into a vector space over $\mathbb{R}$ and is denoted as T_pM (the tangent space at p). An element of T_pM is called a tangent vector at p.
Proposition (how to prove?): Suppose $p \in M$, $v \in T_pM$, and $f, g \in \mathcal{C}^\infty(M)$. Then, if f is constant, $v(f) = 0$. Moreover, if $f(p) = g(p) = 0$, then $v(fg) = 0$.

We need to connect T_pM to $T_p\mathbb{R}^n$ using coordinate charts. To this end, we need to know how smooth maps change tangent spaces. For maps between $\mathbb{R}^n$, tangent space changes can be computed using the derivative matrix which is a linear map from $\mathbb{R}^n$ to itself. Unfortunately, the notion of a linear map between manifolds makes no sense. The best we can hope for is a linear map between tangent spaces.

Let M, N be manifolds (with or without boundary), $F : M \to N$ be a smooth map. The pushforward/differential $(F_*)_p : T_pM \to T_{F(p)}N$ of F at p is defined as the derivation $(F_*)_p(v)(f) = v(f \circ F)$. (why is it a derivation?)
Properties: F_* is linear, $((G \circ F)_*)_p = (G_*)_{F(p)} \circ (F_*)_p$, $I_* = I$, and if F is a diffeo, then $(F_*)_p^{-1} = ((F^{-1})_*)_{F(p)}$.

Some more properties (let M be a manifold with or without boundary):

- Locality: Suppose $v \in T_pM$. If $f, g \in \mathcal{C}^\infty(M)$ agree on a neighbourhood U of p, then $v(f) = v(g)$.
  Proof: Let $\rho : M \to \mathbb{R}$ be a bump such that $\rho = 1$ on $V \subset U$ and $supp(\rho) \subset U$. Then $\rho(f - g) = 0$ on M. Now $0 = v(\rho(f - g)) = 0 + \rho(p)v(f - g) = v(f - g)$. $\square$

- Identification for open submanifolds: Let $U \subset M$ be an open subset. Then $(i_*)_p : T_pU \to T_pM$ is an isomorphism for all $p \in U$.
  Proof: 1-1: If $(i_*)_p(v) = 0$, then whenever $f \in C^\infty(M)$, and $v(f\|_U) = 0$, then suppose $g \in C^\infty(U)$. Let $\rho : M \to \mathbb{R}$ be a bump function equal to 1 in a neighbourhood of p and $supp(\rho) \subset U$. Thus $\rho g : M \to \mathbb{R}$ agrees with f in a neighbourhood of p. Hence $v(\rho g) = 0 = v(g)$ because ρg agrees with g in a neighbourhood of p. Thus $v = 0$.
  Onto: Let $w \in T_pM$. Given $f \in C^\infty(U)$, define $v(f) = w(\rho f)$. We claim that $w(\rho_1 f) = w(\rho_2 f)$ if ρ_1, ρ_2 are two bump functions around p. Indeed, $w((\rho_1 - \rho_2)f) = 0$ because $(\rho_1 - \rho_2)f$ agrees with the constant function zero in a neighbourhood of p. Thus, $v(fg) = w(\rho fg) = w(\rho^2 fg) = w(\rho f)g(p) + w(\rho g)f(p)$. Thus $v \in T_pU$ and $i_*(v)(f) = v(f|_U) = w(\rho f|_U) = w(\rho f)$. $\square$
  Since this isomorphism is independent of choices, we abuse notation and *identify* T_pU with T_pM without mentioning the same.

- Dimension: If M is an n-dimensional manifold (*without* boundary), then T_pM is n-dimensional. (This is applicable even to interior points on manifolds-with-boundary.)
  Proof: Let (ϕ, U) be a coordinate chart around p. Then $(\phi_*)_p : T_pU = T_pM \to T_{\phi(p)}(\phi(U)) = T_{\phi(p)}\mathbb{R}^n = \mathbb{R}^n$ is an isomorphism. $\square$

Unfortunately, this theorem cannot be directly applied to the boundary points on manifolds-with-boundary. (Because $\mathbb{H}^n$ is not an open subset of $\mathbb{R}^n$.) So what is the dimension of T_pM for a boundary point? Is it n or $n-1$? (Spoiler alert: It is n.)
For any $a \in \partial\mathbb{H}^n$, $(i_*)_a : T_a\mathbb{H}^n \to T_a\mathbb{R}^n$ is an isomorphism.
Proof: 1-1: Let $v \in T_a\mathbb{H}^n$ such that $i_*v = 0$, and $f \in C^\infty(\mathbb{H}^n)$. Let $\tilde{f}$ be a smooth extension to $\mathbb{R}^n$. Now $0 = i_*v(\tilde{f}) = v(\tilde{f} \circ i) = v(f)$.
Onto: Let $w = w^i \frac{\partial}{\partial x^i} \in T_a\mathbb{R}^n$. Let $f \in C^\infty(\mathbb{H}^n)$. Define $\tilde{f}$ as a smooth extension of f to $\mathbb{R}^n$ and $v(f) = w(\tilde{f}) = w^i \frac{\partial \tilde{f}}{\partial x^i}(a) =$ and is hence independent of the choice of $\tilde{f}$ (because of continuity). v is a derivation and hence we are done. $\square$
Corollary: The dimension of T_pM even for manifolds-with-boundary is $dim(M)$.