MA 235 - Lecture 26

1 Recap

1. Examples of orientable and non-orientable manifolds. In particular, hypersurfaces admitting nowhere vanishing non-tangent vector fields and ∂M (two orientations that agree with each other when the dimension of M is even).
2. Change of integrals of top forms under diffeomorphisms.

2 Orientation and integration of top forms

Let ω be a top form on an oriented manifold-with-boundary M that is compactly supported in a chart (U, ϕ). Then $\int_M \omega := \pm \int_{\phi(U)} (\phi^{-1})^* \omega$ depending on whether ϕ is orientation-preserving ($[\phi^*(dx^1 \wedge \ldots)] = [\tilde{\omega}|_U]$ where $\tilde{\omega}$ is an orientation form) or reversing.
Proposition: This definition is independent of (U, ϕ).
Proof: Suppose (V, ψ) is another chart. Then $\int_{\phi(U)} (\phi^{-1})^* \omega = \int_{\phi(U \cap V)} (\phi^{-1})^* \omega$. Now $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is a diffeo. From the above results we are done. □
Def: Let M be an oriented manifold-with-boundary. Let ω be a compactly supported top form. Let U_i be a finite coordinate open cover of $supp(\omega)$ (the charts need not have positive orientation). Let ρ_i be a partition-of-unity subordinate to U_i. Then $\int_M \omega := \sum_i \int \rho_i \omega$. (We allow for negatively oriented charts too to care of the 1-dimensional case.)

Proposition: The above definition is independent of the choice of the partition-of-unity.
Proof: Suppose V_j is another open cover, and ρ'_j is another partition-of-unity. Then $\sum_i \int \rho_i \omega = \sum_i \int \sum_j \rho_i \rho'_j \omega = \sum_i \pm \int_{\phi_i(U_i)} \sum_j (\phi_i^{-1})^* (\rho_i \rho'_j \omega)$ which by linearity of the integral is $\sum_i \sum_j \pm \int_{\phi_i(U_i)} (\phi_j^{-1})^* (\rho_i \rho'_j \omega) = \sum_i \sum_j \pm \int_{\psi_j(V_j)} (\psi_j^{-1})^* (\rho_i \rho'_j \omega) = \sum_j \int_{\psi_j(V_j)} (\psi_j^{-1})^* (\sum_i \rho_i \rho'_j \omega) = \sum_j \int_M \rho'_j \omega$. □
For 0-dimensional oriented manifolds, i.e., a discrete collection of points, the "integral" of a compactly supported function f is defined to be $\sum \pm f(p)$ where the signs are decided by the orientation. If $S \subset M$ is a submanifold, and ω is an $n-1$ form, then $\int_S \omega$ is understood to be with respect to the induced orientation (if an $\vec{N}$ is chosen). Likewise for $\int_{\partial M} \omega$ (with the outward normal).

Properties (can be proven directly):

- Linearity.
- Orientation reversal.
- Positivity.
- Diffeomorphism invariance (upto orientation).

Practically speaking... suppose $D \subset \mathbb{R}^2$ is the unit disc (with orientation $dx \wedge dy$) and $\omega = x^2 dx \wedge dy$ is a smooth 2-form on $\bar{D}$. Then how can we calculate $\int_{\bar{D}} \omega$? The problem is that we have to use a partition-of-unity and such things are practically impossible to integrate explicitly! If we were to do it naively, we would have simply done $\int_{x^2+y^2\leq 1} x^2 dxdy = \int_0^{2\pi} \int_0^1 r^2 \cos^2(\theta) r dr d\theta$.

To relate these two, here is a proposition: Let ω be a compactly supported top form on M. Let $D_1, \ldots, D_k$ be domains of integration in $\mathbb{R}^n$ and $F_i : \bar{D}_i \to M$ be smooth maps that restrict to orientation-preserving diffeos on D_i, $F(D_i) \cap F(D_j) = \phi$, $supp(\omega) \subset F(\bar{D}_1) \cup F(\bar{D}_2) \ldots$. Then $\int_M \omega = \sum_i \int_{D_i} F_i^* \omega$.

Before proving it, note that the identity map does the trick for $D \subset \mathbb{R}^2$ above. Thus $\int_D \omega = \int_{x^2+y^2<1} x^2 dxdy$.
Proof: As above, assume WLOG that ω is supported in a single relatively compact chart (U, ϕ). Let $supp(\omega) \subset V \subset \bar{V} \subset U$ be such that ∂V has measure zero. Note that $\phi(\partial(V \cap F_i(D_i)))$ has measure zero in $\mathbb{R}^n$: Indeed, smooth maps between $\mathbb{R}^n$ and itself take measure zero sets to measure zero sets (why?). Moreover, $\phi(V \cap F_i(D_i))$ cover $\phi(supp(\omega))$ upto measure zero sets and are pairwise disjoint.
Thus $\int_M \omega = \pm \int_{\phi(U)} (\phi^{-1})^* \omega = \pm \sum_i \int_{\phi(U \cap F_i(D_i))} (\phi^{-1})^* \omega = \sum_i \int_{D_i} F_i^* \omega$ (why?) $\square$
Actually, one does not need F_i to extend smoothly to $\bar{D}_i$. Lipschitz (or even weaker - Hölder) extensions are enough.