MA 235 - Lecture 17

1 Recap

1. Tangent bundle.

2. Vector bundles (examples).

3. Smooth sections.

Def: Let V, W be vector bundles over M. A smooth map $T : V \to W$ is called a vector bundle map/morphism if T commutes with the projections, i.e., T takes V_p into W_p for all $p \in M$ and does so linearly. If T is $1-1$ and T^{-1} is also smooth, then T^{-1} is also a vector bundle morphism. Such a T is called an isomorphism between V and W.
Non-example: The Möbius bundle is *not* isomorphic to $S^1 \times \mathbb{R}$. TS^2 is *not* isomorphic to $S^2 \times \mathbb{R}^2$. On the other hand, TS^1 *is* isomorphic to $S^1 \times \mathbb{R}$.
Example: There is always an isomorphism $I : V \to V$ given by $I(v) = v$.
Def: Let $S \subset V$ be a vector bundle such that the inclusion map is an embedding and a $1-1$ vector bundle map. Then S is said to be a subbundle of V.
Let V be a vector bundle over M. Consider the set $V^* = \cup_p V_p^*$. We can make V^* into a smooth vector bundle over M as well (called the dual bundle of V). Indeed, cover M by means of coordinate charts (U_α, x_α) such that V is trivial over U_α. Let $T_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$ be a local trivialisation for V. Consider the sections $s_{i,\alpha} = T_\alpha^{-1}(e_i)$. Take the dual smoothly varying basis $(s_\alpha^*)^i$ defined as $(s_\alpha^*)^i(p)(s_{\alpha,j}(p)) = \delta_j^i$. Consider the map $L_\alpha : U \times \mathbb{R}^r \to V^*$ given by $L_\alpha(p, \vec{v}) = v_i (s_\alpha^*)^i$. This map is a bijection. Define a topology on V^* by the same construction as for TM. By the same reasoning, V^* with such a topology is Hausdorff and second-countable. Similar to TM, it has a countable collection of coordinate charts making it into a smooth manifold. These charts are induced from L_α which actually make them local trivialisations.

This construction applied to TM produces a vector bundle T^*M known as the cotangent bundle. The smooth sections of T^*M are called 1-form fields.

2 Vector fields and flows

Recall that the point of defining vector fields/tangent bundle was to model the flow of fluids along a manifold.
Def: Let M be a smooth manifold with or without boundary. Let $J \subset \mathbb{R}$ be an interval containing 0 in its interior. Let X be a smooth vector field on M. An integral

curve for X starting at $p \in M$ is a smooth path $\gamma : J \to M$ such that $\gamma(0) = p$ and $\gamma'(t) = X(\gamma(t)) \, \forall \, t \in J$.

Examples:

- The constant vector field in $\mathbb{R}^n$: Let $X = c^i \frac{\partial}{\partial x^i}$ on $\mathbb{R}^n$. The integral curve starting at $\vec{p}$ is $\vec{p} + \vec{c}t$ (why?) Note that if we fix t, then $\vec{p} \to \vec{p} + \vec{c}t$ is a diffeomorphism of $\mathbb{R}^n$!

- Rotational vector field in $\mathbb{R}^2$: $X = (y, -x)$. Now $\frac{dx}{dt} = y, \frac{dy}{dt} = -x$. Thus $\frac{d^2x}{dt^2} = -x, \frac{d^2y}{dt^2} = -y$. Hence $x = A\cos(t) + B\sin(t), y = x' = -A\sin(t) + B\cos(t)$. At $t = 0$, $x_0 = A, y_0 = B$. Fixing t, $(x_0, y_0) \to (x_0\cos(t) + y_0\sin(t), -x_0\sin(t) + y_0\cos(t))$ is a diffeo of $\mathbb{R}^2$!

- Incomplete vector field in $\mathbb{R}^2 - \{0\}$: Consider the constant vector field $X = \frac{\partial}{\partial x}$ in $\mathbb{R}^2 - \{0\}$. The integral curve for say $(-1, 0)$ does not exist at $t = 1$! Such a vector field (that has at least one integral curve that does not exist for all time) is called an incomplete vector field.

- A vector field that blows up in finite-time in $\mathbb{R}^2$: $X = x^2 \frac{\partial}{\partial x}$. Note that $\frac{dx}{dt} = x^2, \frac{dy}{dt} = 0$. Hence, $x = \frac{x_0}{1 - x_0 t}, y = y_0$. In other words, it blows up in finite time! (Incomplete vector field.)

- A compactly supported vector field in $\mathbb{R}^n$: Let X be any smooth vector field. Let ρ be a smooth compactly supported function. Then ρX is a compactly supported vector field. For any starting point outside the support, the integral curve is a constant!

Theorem (Existence of integral curves): Let X be a smooth vector field on a manifold M (without boundary). For every $p \in M$, there exists a neighbourhood $U_p \subset M$ and $\epsilon_p > 0$ such that for every $q \in U_p$, there is a unique smooth integral curve $\gamma : (-\epsilon_p, \epsilon_p) \to M$ for X starting at q.
This theorem follows almost immediately from the existence/uniqueness/smooth-dependence-on-initial-parameters theorem for (time-independent) systems of ODE. That theorem is proven by rewriting the system as an integral equation and using an iterative method and the contraction mapping principle. For uniqueness and smoothness, one needs to put in more effort (Gronwall's inequality). The ϵ_p can be finite and U_p need not be all of M.
Let $\gamma : J \to M$ be a smooth integral curve. Then $\tilde{\gamma} : \tilde{J} \to M$ given by $\tilde{\gamma}(t) = \gamma(at)$ where $a \in \mathbb{R}$ is an integral curve for aX starting at p. $\tilde{\gamma} : J - b \to M$ given by $\tilde{\gamma}(t) = \gamma(t + b)$ is an integral curve for X. Suppose M, N are smooth manifolds, $F : M \to N$ is a smooth map, and X is a smooth vector field on M. Unfortunately, there need not be a smooth vector field on N that is a "pushforward" of X (why?) If F is a diffeo, then one can talk of pushforwards of vector fields. However, if Y is a smooth vector field on N such that $Y_{F(p)} = (F_*)_p(X_p)$ for all $p \in M$ (then Y and X are said to be F-related), then whenever γ is an integral curve for X, $F \circ \gamma$ is an integral curve for Y (why?)

Theorem (Flows of compactly supported vector fields): Every smooth compactly supported vector field X is complete. In particular, any smooth vector field on a compact manifold is complete.
Proof: Let p be a starting point. Let T be the supremum of all ϵ such that the integral curve exists on $(-\epsilon, \epsilon)$. If $T < \infty$, then firstly p is within the support of X (why?). Secondly, consider a sequence $t_n \to T$. Then both the sequences $q_n = \gamma(-t_n), r_n = \gamma(t_n)$ are within a compact set - in fact, the support of X (why?). Hence there are convergent subsequences (that we still call q_n, r_n abusing notation). So $q_n \to q$ and $r_n \to r$. Now integral curves exist with q, r as starting points for some time. Hence, there are two integral curves starting at q_n, r_n (for some large n). By uniqueness, they coincide. Hence, the integral curve can be smoothly extended past T. A contradiction. $\square$

Theorem: If X is compactly supported, then $\theta : \mathbb{R} \times M$ given by $\theta_t(p) = \gamma(t)$ where γ is the integral curve of X starting from p (this map is called the time-t flow of X) is smooth and a diffeomorphism for each fixed t. Moreover, $\theta_{t+s} = \theta_t \circ \theta_s$ and $\theta_{-t} = \theta_t^{-1}$. (Such a collection of diffeomorphisms is a called a one-parameter group.)
Proof: The smoothness follows from local-smooth-dependence-on-initial-conditions. Now if we prove that $\theta_{-t} = \theta_t^{-1}$, then θ_t is a diffeo. Indeed, θ_0 is identity. If we prove that $\theta_{t+s} = \theta_t \circ \theta_s$, we are done. This follows from uniqueness (why?) $\square$