MA 235 - Lecture 18

1 Recap

1. Dual bundle.
2. Integral curves (existence, uniqueness, smooth dependence).
3. Compactly supported vector fields are complete and define a one-parameter flow.

2 Vector fields and flows

Theorem: For every smooth connected manifold M (without boundary), and points $p, q \in M$, there exists a diffeomorphism that takes p to q. Proof: The set of all points that can be obtained from p by diffeomorphisms of M is non-empty (why?) If we prove that it is open and closed, we are done (why?)
Openness: Suppose q can be obtained from p by a diffeo. We shall prove that there exists a diffeo that takes q to any point nearby but fixes p. Composition will then do the job. Indeed, take a constant vector field in coordinates and cut it off by a bump function. The flow of this compactly supported vector field does the job.
Closedness: Suppose $q_n \to q$. There is again a neighbourhood of q, consisting of points that can be obtained from q by diffeos fixing p. q_n lies in the neighbourhood for some n. Again, composition does the job. □

Another application of flows: What is an example of a local smooth vector field near a point? The simplest one is $\frac{\partial}{\partial x^1}$ for some coordinate x^1. A related physical question: Following a small paper boat in a river, can we somehow get a sense of the time elapsed? Of course, the farther the paper boat gets, the more the time has elapsed. However, what if the boat is placed at a point where the river isn't moving?
Theorem: Let X be a smooth vector field on a smooth manifold M (without boundary). Suppose $X(p) \neq 0$. Then there exists a neighbourhood around p and a coordinate chart s^i such that $X = \frac{\partial}{\partial s^1}$ in that neighbourhood.
In a sense, s^1 functions as a "time coordinate".
Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that there is an integral curve on $(-\epsilon, \epsilon)$ starting from any $q \in U$. Consider the map $F : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \times V \to M$ (where V is some neighbourhood of the origin) given by $F(s, x^2, \ldots, x^n) = \gamma(s)$ where γ is the integral curve starting at $(0, x^2, \ldots, x^n)$. By the smooth dependence part of the existence theorem, F is a smooth map. Assuming V is small enough, we can assume that the image of F lies in U.

The derivative of F is $(s=0,0,\ldots,0)$ is $DF = \begin{bmatrix} X^1(p) & 0 & 0 & \ldots \\ X^2(p) & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$ (why?), which is invertible (why?) Thus by IFT, F is a local diffeo and hence $(s, x^2, \ldots, x^n)$ is a new coordinate chart around p. In this chart, the integral curve starting at $(0, x^2, \ldots, x^n)$ is $\gamma(t) = (t, x^2, \ldots, x^n)$. Thus, $\gamma' = (1, 0, \ldots, 0)$ and $X = \frac{\partial}{\partial s}$. $\square$

Collar neighbourhood theorem (proof omitted): If M is a smooth manifold-with-boundary whose boundary is compact, there is a neighbourhood of ∂M that is diffeomorphic to a "collar" $[0,1) \times \partial M$ such that ∂M goes to $\{0\} \times \partial M$.
The point is that using this theorem one can define the "double" of a manifold-with-boundary. One can also define a connected sum of two manifolds by removing spheres and "gluing" them.

3 Lie bracket

Suppose we have two vector fields X, Y that are linearly independent at p. Can we say that there is a neighbourhood and a coordinate chart s^i such that $X = \frac{\partial}{\partial s^1}$ and $Y = \frac{\partial}{\partial s^2}$? If there was such a chart, the coordinate "axes" should have been obtained by flowing these vector fields. But how do we know that if we around a "coordinate square", we will come back to the same point? In other words, $\gamma(s,p)$ and $\psi(t,p)$ are integral curves of X, Y resp. starting at p, then how do we know that $\gamma(-s, \psi(-t, \gamma(s, \psi(t,p)))) = p$? At least, is $\frac{\partial^2 \gamma(-s,\psi(-t,\gamma(s,\psi(t,p))))}{\partial s \partial t}|_{s=t=0} = 0$?
$\frac{\partial \gamma^i(-s,F(s,t))}{\partial s}|_{s=0} = -\frac{d\gamma^i}{ds}|_{s=0} + \frac{\partial \gamma^i}{\partial x^j}\frac{\partial F^j}{\partial s}|_{s=0} = -X^i(F(0,t)) + \frac{\partial F^i}{\partial s}|_{s=0}$. Now $\frac{\partial F^j}{\partial s}|_{s=0} = \frac{\partial \psi^j}{\partial x^k}|_{s=0}\frac{\partial \gamma^k}{\partial s}|_{s=0} = \frac{\partial \psi^j}{\partial x^k}(-t, \psi(t,p))X^k(\psi(t,p))$. Taking one more derivative, we get $\frac{\partial^2 \gamma(-s,\psi(-t,\gamma(s,\psi(t,p))))}{\partial s \partial t}|_{s=}$ $-\frac{\partial X^i}{\partial x^j}(p)\frac{\partial F^j}{\partial t}|_{t=0} + \frac{\partial}{\partial t}\left(\frac{\partial \psi^i}{\partial x^k}(-t, \psi(t,p))X^k(\psi(t,p))\right)_{t=0}$, which is $-\frac{\partial Y^i}{\partial x^k}(p)X^k(p) + \frac{\partial X^i}{\partial x^k}(p)Y^k(p)$.

This last expression actually defines a vector field (why?) Is there an coordinate-invariant way of defining this vector field?
Def: Let X, Y be smooth vector fields on a manifold M. Then $[X,Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$ is a vector field on M called the Lie bracket of X and Y.
Lemma (proof by calculation): The Lie bracket genuinely defines a vector field whose components are given above.
Properties:

- $[X,Y] = -[Y,X]$.
- $[X,Y]$ is multi-linear in X, Y.
- It is not associative! That is $[X,[Y,Z]] \neq [[X,Y],Z]$! Indeed, $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ (Jacobi's identity).

Any vector space equipped with such a "bracket" is called a Lie algebra. The space of smooth vector fields is an example. If X, Y are coordinate vector fields, they Lie-commute (why?).

Conversely, Theorem (proof omitted): If $X^1, X^2, \ldots, X^k$ are smooth Lie-commuting vector fields that are linearly independent at p, there is a neighbourhood and a coordinate chart such that $X^i = \frac{\partial}{\partial x^i}$.