MA 235 - Lecture 27

1 Recap

1. Integration of top forms (definition).

2. Practical calculation by covering with charts from $\mathbb{R}^n$ upto measure zero. Here we can do better than the theorem stated in the last class: Let ω be a compactly supported top form on M. Let $D_1, \ldots, D_k$ be bounded domains of integration in $\mathbb{R}^n$ and $F_i : \bar{D}_i \to M$ be continuous maps that restrict to orientation-preserving diffeos on D_i, $F_i(D_i) \cap F_j(D_j) = \phi$, $supp(\omega) \subset F_1(\bar{D}_1) \cup F_2(\bar{D}_2) \ldots$, $F_i(\bar{D}_i) - F(D_i)$ is of measure zero in the manifold for all i. Then $\int_M \omega = \sum_i \int_{D_i} F_i^* \omega$ if the right-hand-side is Lebesgue integrable.
   Proof: As before, it is enough to assume that ω is compactly supported in a chart U. Thus, we have reduced our problem to an open subset in $\mathbb{R}^n$ (after throwing out the measure zero boundary in $\mathbb{H}^n$). Since the boundaries are of measure zero, the integral is $\sum_i \int_{F(D_i)} \omega$. Now for each summand, since $F(D_i)$ is an open subset of $\mathbb{R}^n$, it can be exhausted by submanifolds-with-boundary $K_{N,i}$. By the dominated convergence theorem, $\int_{F(D_i)} \omega = \lim_{N\to\infty} \int_{K_{N,i}} \omega = \lim_{N\to\infty} \int_{F_i^{-1}(K_{N,i})} F_i^* \omega = \lim_{N\to\infty} \int_{F_i^{-1}(K_{N,i})} F_i^* \omega = \int_{D_i} F_i^* \omega$ by the dominated convergence theorem.

2 Stokes' theorem

Theorem: Let M be a smooth oriented n-manifold-with-boundary (where the boundary has the induced orientation from outward vector fields). Let ω be a compactly supported $n-1$ form on M. Then $\int_M d\omega = \int_{\partial M} \omega$. (In particular, if $\partial M = \phi$, then $\int_M d\omega = 0$.)
Before we proceed to the proof, suppose M is a domain in $\mathbb{R}^2$, and $\omega = Pdx + Qdy$, then $\int_M d\omega = \int_M (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dV$ and $\int_{\partial M} \omega = \int_{\partial M} (Pdx + Qdy)$. If ∂M can be parametrised as $\gamma : [0,1] \to \partial M$ where γ is a smooth simple closed curve such that $\gamma' \neq 0$, then by the above result, $\int_{\partial M} \omega = \int_{(0,1)} \gamma^* \omega = \int_0^1 (P\frac{dx}{dt} + Q\frac{dy}{dt}) dt$. (A small point: the orientation of ∂M corresponds to travelling anticlockwise (why?)) Thus we have proven Green's theorem. (Extends to the multiply connected case.)

Proof: Cover the support of ω by finitely many charts (interior or boundary) U_i. Let ρ_i be a partition-of-unity subordinate to this cover. Then $\int_M d\omega = \sum_i \int_M d(\rho_i \omega)$.

Thus, if we prove Stokes for $\rho_i \omega$, i.e., for forms that are compactly supported in a chart (interior or boundary), then we are done. (why?) So assume wlog that ω is compactly supported in a chart (U, ϕ). Wlog, ϕ is positively oriented (why?) Thus $\int_M d\omega = \int_{\phi(U)} d(\phi^{-1})^* \omega$. Therefore, it is enough to assume that M is $\mathbb{H}^n$ or $\mathbb{R}^n$.

We have two cases:

- $M = \mathbb{R}^n$: Let $\omega = \omega_i dx^1 \ldots dx^{i-1} \wedge \hat{dx^i} \wedge \ldots$. Now $\int_{\mathbb{R}^n} d\omega = \int_{\mathbb{R}^n} \sum_i \frac{\partial \omega_i}{\partial x^i}(-1)^{i-1} dx^1 \wedge \ldots$. This latter expression is 0. (Why?)
- $M = \mathbb{H}^n$: Assume that the support is in $[-A, A]^{n-1} \times [0, A]$. Now $\int_{\mathbb{H}^n} d\omega = \int_{-A}^{A} \ldots \int_{-A}^{A} \int_0^A \sum_i \frac{\partial \omega_i}{\partial x^i}(-1)^{i-1} dx^n \ldots = \int_{\mathbb{R}^{n-1}} (-1)^n \omega_n(x, 0) + 0$ (why?) Now the boundary $\mathbb{R}^{n-1}$ has orientation form $dx^1 \wedge dx^2 \ldots (-\frac{\partial}{\partial x^n}, \ldots) = (-1)^n dx^1 \wedge \ldots$. Thus the last integral equals $\int_{\partial \mathbb{H}^n} \omega$. $\square$

Consequences of Stokes:

- All the classical theorems (Divergence, Stokes, Green) are special cases.
- If S is a compact oriented submanifold of a smooth manifold M, and ω is a closed k-form on M, such that $\int_S \omega \neq 0$, then ω is NOT exact, and S is NOT the boundary of a submanifold (why?)
- Thus, $\omega = \frac{xdy - ydx}{x^2+y^2}$ is closed but not exact.
- Suppose M is an oriented compact smooth manifold with boundary. There is *no* smooth retraction of M onto its boundary: Recall that $r : M \to \partial M$ is a retract if r is identity on ∂M. If there is a retract, then suppose ω is an orientation form on ∂M. Then $r^* \omega$ is a smooth $n-1$ form on M that restricts to ω on ∂M. Now $\int_M dr^* \omega = \int_{\partial M} \omega > 0$. However, $d(r^* \omega) = r^*(d\omega) = 0!$ $\square$

Example of Stokes: Let M be the orientable compact 3-manifold-with-boundary $\{(x, y, z, w) \in \mathbb{R}^4 \mid w \geq 0, x^2 + y^2 + z^2 + w^2 = 1\}$. Let $\omega = z^2 dx \wedge dy + x^2 dy \wedge dz + dx \wedge dw$ on $\mathbb{R}^4$. Then we need to choose an orientation on M and verify the generalised Stokes' theorem for ω $\left(\text{that is, } \int_M d\omega = \int_{\partial M} \omega\right)$ by explicitly calculating the left and right hand sides:
Consider the smooth parametrisation $\alpha(u_1, u_2, u_3) = (u_1, u_2, u_3, \sqrt{1 - u_1^2 - u_2^2 - u_3^2})$ defined on the open unit ball in $\mathbb{R}^3$ to the interior of M. This map is smooth (by the Chain rule), $1-1$ (trivially), onto the interior (trivially), and the inverse is a projection which is smooth. Moreover, $D\alpha(p, q, r) = (p, q, r, -\frac{u_1 p + u_2 q + u_3 r}{\sqrt{1 - u_1^2 - u_2^2 - u_3^2}}$ which is clearly $1-1$. Choose the orientation that this parametrisation belongs to.
$d\omega = 2z dz \wedge dx \wedge dy + 2x dx \wedge dy \wedge dz + 0 = 2(x + z) dx \wedge dy \wedge dz$.
Now $\alpha^*(d\omega) = 2(u_1 + u_2) du_1 \wedge du_2 \wedge du_3$.
Since the parametrisation covers all of M except for the boundary, which is of measure 0 because it is a finite union of images of the $u_3 = 0$ plane in $\mathbb{H}^3$ which has measure 0 in $\mathbb{H}^3$, we can calculate the integral as the improper integral $\int_{u_1^2+u_2^2+u_3^2<1} \alpha^*(d\omega) = \int_{u_1^2+u_2^2+u_3^2<1} 2(u_1 + u_2)$ which is in fact a Riemann integral because the integrand is bounded and continuous and the domain is a compact rectifiable one (the boundary of the domain is a

sphere which is a union of two graphs and hence has measure zero). By Fubini's theorem this integral is 0 .
Consider the parametrisations of a part of M given by $\beta_{\pm}(v_1, v_2, v_3) = (\pm\sqrt{1 - v_1^2 - v_2^2 - v_3^2}, v_1, v_2, v_3)$ from the the open unit ball intersect $v_3 \geq 0$ (an open subset of $\mathbb{H}^3$) to M that cover a neighbourhood of a part of the boundary. These maps are smooth, $1-1$, and the inverses are projections. Moreover, just as before, $D\beta_{\pm}$ are $1-1$. The part of the boundary this is missed is the image of $v_2^2 + v_3^2 = 1$ which has measure 0 in $\mathbb{R}^2$ because it is a union of two graphs.
The maps $\phi_{\pm} = \alpha^{-1} \circ \beta_{\pm}(v_1, v_2, v_3) = (\pm\sqrt{1 - v_1^2 - v_2^2 - v_3^2}, v_1, v_2)$ satisfy $\det(D\phi_{\pm})(0, 0, 1/\sqrt{2}) = \mp 1$ (and hence the signs stay the same throughout the domains of definition because the domains are connected). This means that β_+ is not compatible with α and β_- is so. Therefore we change β_- to $\tilde{\beta}_-(v_1, v_2, v_3) = \beta_-(-v_1, v_2, v_3)$. Now the $\tilde{\beta}_+(v_1, v_2, 0) := \beta_+(v_1, v_2, 0), \tilde{\beta}_-(v_1, v_2, 0)$ have the correct orientations for Stokes' theorem because the dimension of M is odd.
They parametrise the boundary upto measure zero. Moreover, $\tilde{\beta}_{\pm}^* \omega = v_2^2 d(\pm\sqrt{1 - v_1^2 - v_2^2}) \wedge d(\pm v_1) + (1 - v_1^2 - v_2^2) d(\pm v_1) \wedge dv_2 = \left(\frac{v_2^3}{\sqrt{1 - v_1^2 - v_2^2}} \pm (1 - v_1^2 - v_2^2) \right) dv_1 \wedge dv_2$.
Hence by the theorems above, as improper integrals,

$$\int_{\partial M} \omega = \int\int_{v_1^2 + v_2^2 < 1} \left(\frac{v_2^3}{\sqrt{1 - v_1^2 - v_2^2}} + (1 - v_1^2 - v_2^2) \right) + \int\int_{v_1^2 + v_2^2 < 1} \left(\frac{v_2^3}{\sqrt{1 - v_1^2 - v_2^2}} - (1 - v_1^2 - v_2^2) \right)$$

Now the integral $\int\int_{v_1^2 + v_2^2 < 1} |\frac{v_2^3}{\sqrt{1 - v_1^2 - v_2^2}}|$ exists in the Lebesgue sense. Indeed, the integrand is at most $\frac{1}{\sqrt{1 - v_1^2 - v_2^2}}$. By the monotone convergence theorem, $\int\int_{v_1^2 + v_2^2 < 1} \frac{1}{\sqrt{1 - v_1^2 - v_2^2}} = \lim_{n \to} \int\int_{U_n = v_1^2 + v_2^2 < (1 - 1/n)^2} \frac{1}{\sqrt{1 - v_1^2 - v_2^2}}$ which exists by the change of variables formula for instance.
Now the above improper integrals are equal to (by linearity)

$$\int\int_{v_1^2 + v_2^2 < 1} \frac{2v_2^3}{\sqrt{1 - v_1^2 - v_2^2}},$$

which by the change of variables formula is

$$\lim_{n \to \infty} \int\int_{(0, 1 - 1/n) \times (0, 2\pi)} \frac{2r^4 \cos^3(\theta) \sin^3(\theta)}{\sqrt{1 - r^2}}$$

which is given by Fubini's theorem as 0. □

3 Looking beyond

- Differential topology (When are two manifolds homeomorphic but not diffeomorphic?)
- Riemannian geometry (distances, angles, curvature, congruence/isomorphism, finding the best metric, etc)

- Geometric analysis (PDE on manifolds)
- Symplectic geometry (Classical mechanics on manifolds)
- Algebraic geometry (Zeroes of polynomials)
- Arithmetic geometry and number theory (Modular forms and elliptic curves for instance)
- Applications (Protein folding, control theory, general relativity, string theory, statistical mechanics, etc)