HW 1(to be submitted in the second week of January)

1. Let $f(x, y) = \frac{xy^2}{x^2+y^4}$ if $x \neq 0$ and $f(0, y) = 0$. Prove that f is directionally differentiable along all directions at $(0, 0)$ but is not differentiable at $(0, 0)$.

2. Let $f(x, y) = ||x| - |y|| - |x| - |y|$. Prove that the partials exist and f is continuous at $(0, 0)$ but it isn't differentiable there.

3. Prove that a symmetric real 2×2 matrix S is positive-definite iff the diagonal entries of S are positive and $\det(S) > 0$.

4. Find the critical points of $f(x, y) = 4x^2 + 9y^2 + 8x - 36y + 24$ and determine their nature.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a twice-differentiable function whose second partials are continuous everywhere. We shall analyse a continuous version of gradient descent in this question. Assume that a twice-differentiable solution (whose second derivative is continuous) $\vec{r}(t) : [0, \infty)$ exists to the equation $\frac{d\vec{r}}{dt} = -\nabla f(\vec{r}(t))$ with $\vec{r}(0) = \vec{r}_0$. Assume that $Hess(f) - cI$ is positive-definite for some $c > 0$ and $Hess(f)$ is the Hessian matrix of f. Assume that a global minimum for f exists and is unique.

   (a) Prove that $\frac{df}{dt} = -|\nabla f|^2$.

   (b) Prove that $\frac{d^2 f}{dt^2} = 2 \sum_{i,j} f_i f_{ij} f_j$.

   (c) As a consequence, prove that $\frac{d^2 f}{dt^2} \geq 2c|\nabla f|^2$ and that $\frac{d|\nabla f|^2}{dt} \leq -2c|\nabla f|^2$.

   (d) Prove that $|\nabla f|^2(\vec{r}(t)) \leq |\nabla f|^2(\vec{r}_0)e^{-2ct}$.

   (e) Prove that $|\vec{r}(t) - \vec{r}(s)| \leq A(e^{-cs} - e^{-ct})|$ for some A and for all $s, t \geq 0$.

   (f) Prove that $\vec{a} = \lim_{t \to \infty} \vec{r}(t)$ exists.

   (g) Prove that $\vec{a}$ is the global minimum of f.