MA 235 - Lectures 10 - 14

1 Tangent spaces

Examples of tangent spaces:

- Let V be a f.d normed vector space treated as a smooth manifold. Consider the map $D_{a,v}f = \frac{df(a+tv)}{dt}$. This map gives an isomorphism of V to T_aV that commutes with linear maps to other vector spaces (what does this mean and why?)
- Thus we can canonically identify V with T_aV. Moreover, if $M \subset V$ is an open submanifold, then $T_aM = T_aV = V$. Thus $T_aGL(n,\mathbb{R}) = M(n,\mathbb{R})$.
- Let $M_1, M_2, \ldots, M_k$ be smooth manifolds (without boundary). Then $\alpha_p : T_p(M_1 \times M_2 \ldots) \to T_pM_1 \times T_pM_2 \ldots$ given by $\alpha_p(v) = ((\pi_1)_*(v), (\pi_2)_*(v), \ldots)$ is an isomorphism.

Proposition: Let M be a smooth n-manifold with or without boundary, and $p \in M$. For any chart (U, x^i) around p, the (pushforwards of) the coordinate vectors $\frac{\partial}{\partial x^i}$ form a basis for T_pM, i.e., If $f \in C^\infty(M)$, then $v(f) = v^i \frac{\partial f \circ \phi^{-1}}{\partial x^i}(\phi(p))$. As always, we abuse notation and drop the ϕ. So $v(f) = v^i \frac{\partial f}{\partial x^i}(p)$.
The vectors $\frac{\partial}{\partial x^i}$ are called a coordinate basis for T_pM. Since the map $v \to D_{p,v}$ is an isomorphism in $\mathbb{R}^n$, these vectors can also be identified with $e_1 = (1, 0, 0 \ldots), \ldots$. The components of v in a coordinate chart (U, x^i) are $v^i = v(x^i)$.
Let $F : U \subset \mathbb{R}^m \to V \subset \mathbb{R}^n$ be a smooth map. Then $F_*(\frac{\partial}{\partial x^i})(f) = \frac{\partial (f \circ F)}{\partial x^i}(p) = \frac{\partial f}{\partial y^j}(F(p))\frac{\partial F^j}{\partial x^i}(p)$. In other words, $F_* \frac{\partial}{\partial x^i} = \frac{\partial F^j}{\partial x^i}\frac{\partial}{\partial y^j}$. Thus if v is treated as column vector $\vec{v}$ with components v^i, then F_*v is a column vector obtained by $[DF]\vec{v}$. The *same* formula (with abuse of notation)holds for $F : M \to N$ and (U, x^i), (V, y^j) are coordinates around $p, F(p)$.

Suppose (U, x), $(V, \tilde{x})$ are two coordinate charts around $p \in M$. Suppose $v \in T_pM$. So (abusing notation) $v = v^i \frac{\partial}{\partial x^i}$ and $v = \tilde{v}^j \frac{\partial}{\partial \tilde{x}^j}$. How are the v^i and $\tilde{v}^j$ related? Note that $\tilde{v^j} = v(\tilde{x}^j) = v^i \frac{\partial \tilde{x}^j}{\partial x^i}$.
Example: Consider the polar coordinates (r, θ) and the Cartesian coordinates (x, y). What is the vector $\frac{\partial}{\partial r} + 2\frac{\partial}{\partial \theta}$ in terms of $\hat{i} = \frac{\partial}{\partial x}$ and $\hat{j} = \frac{\partial}{\partial y}$? It is $\frac{\partial x}{\partial r}\hat{i} + \frac{\partial y}{\partial r}\hat{j} + 2\frac{\partial x}{\partial \theta}\hat{i} + 2\frac{\partial y}{\partial \theta}\hat{j}$.
Example (Caution!): Let $\tilde{x} = x$, $\tilde{y} = y + x^3$. Let $p = (1, 0)$ in (x, y) coordinates. Is $\frac{\partial}{\partial x}|_p = \frac{\partial}{\partial \tilde{x}}|_p$? $\frac{\partial}{\partial x} = \frac{\partial \tilde{x}}{\partial x}\frac{\partial}{\partial \tilde{x}} + \frac{\partial \tilde{y}}{\partial x}\frac{\partial}{\partial \tilde{y}}$ which at p is $\frac{\partial}{\partial \tilde{x}} + 3\frac{\partial}{\partial \tilde{y}}$.

Recall that $i : S^n \to \mathbb{R}^{n+1}$ is smooth. Thus $i_* : T_pS^n \to T_pR^{n+1}$ is a linear map. In coordinates: Consider the stereographic charts $U_\pm$. For instance, on U_+, $i(z^1 = \frac{x^1}{1-x^{n+1}}, \ldots, z^n = \frac{x^n}{1-x^{n+1}}) = (x^1 = \frac{2z^i}{1+\sum_j (z^j)^2}, x^2, \ldots, x^{n+1} = \frac{\sum_j (z^j)^2-1}{\sum_j (z^j)^2+1})$. In these coordinates, $i_* \frac{\partial}{\partial z^i} = \frac{\partial x^j}{\partial z^i}\frac{\partial}{\partial x^j}$. It can be easily seen that i_* is $1-1$ and that its image is precisely the usual tangent plane at p.

Another definition of the tangent space (a physicist's definition): Let M be a manifold (with or without boundary). Consider the set $\mathcal{S}$ of all the coordinate charts (U, x) containing p. For every $(U, x) \in \mathcal{S}$, consider the vector space $V_{U,x} = \mathbb{R}^n$, i.e., consider the disjoint union of $\mathbb{R}^n$ over U, x. Define a relation $\sim$ on this disjoint union as $v \in V_{U,x} \sim w \in V_{W,y}$ if $v^i = w^j \frac{\partial x^i}{\partial y^j}(p)$. This relation is an equivalence relation (why?) The set of equivalence classes is defined to be $T_p\tilde{M}$. It is a vector space (how?). Suppose $F : M \to N$ is a smooth map, define $\tilde{F}_*([v]) = [DFv]$. Consider the (choice-free/canonical) map $F : T_pM \to T_p\tilde{M}$ given by $v \to [v^i]$. This map is a linear isomorphism that commutes with pushforwards (HW).

Velocities of paths: Given an interval $J \subset \mathbb{R}$ and a smooth manifold (with or without boundary) M, a smooth path passing through $p \in M$ is a smooth function $\gamma : J \to M$ such that $\gamma(t_0) = p$ for some $t_0 \in J$. (Typically, a curve is the *image* of a path. Warning: Lee calls paths as curves.) Note that the $T_tJ = \mathbb{R}$ for every $t \in J$. The *velocity* of a smooth path at t_0 is $\gamma'(t_0) = (\gamma_*)_{t_0}(\frac{d}{dt}) \in T_pM$. (One also denotes it by various other symbols.) It acts on smooth functions as $\gamma'(t_0)(f) = (f \circ \gamma)'(t_0)$. Suppose (U, x^i) is a coordinate chart around p, $\gamma'(t_0) = \frac{d\gamma^i}{dt}(t_0)\frac{\partial}{\partial x^i}$, i.e., $\gamma'(t_0)(f) = \frac{\partial f}{\partial x^i}(p)\frac{d\gamma^i}{dt}(t_0)$.
Proposition: Every $v \in T_pM$ is the velocity of some smooth path in M passing through p.
Proof: Choose a chart (U, x) centred at p. Now $v = v^i \frac{\partial}{\partial x^i}$. Choose the smooth path $\gamma(t) = t(v^1, \ldots, v^n)$ (abusing notation). Also the domain of γ depends on whether we are dealing with a boundary point or an interior point. Clearly $\gamma'(0) = v$. □Composition (trivial): Let $F : M \to N$ be a smooth map and $\gamma : J \to M$ be a smooth path. Then the velocity of $F \circ \gamma$ at t_0 is $F_*(\gamma'(t_0))$.
Computing the differential: Suppose $F : M \to N$ is smooth and $v \in T_pM$. Then $F_*v = (F \circ \gamma)'(0)$ where $\gamma(0) = p, \gamma'(0) = v$.

Basically, all tangent vectors are velocity vectors of smooth paths. We can turn this around to define tangent vectors. Consider the relation $\sim$ between smooth paths $\gamma : J \to M$ where $0 \in J$ and $\gamma(0) = p$: $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for any real-valued smooth function defined on a neighbourhood of p. This relation is an equivalence relation (why?). V_pM is defined to be the set of equivalence classes. If $F : M \to N$ is a smooth map, then $F_*[\gamma] = [F \circ \gamma]$. The velocity of a smooth path γ is simply $[\gamma]$. Defining a vector space structure isn't easy. The simplest way is: Consider the map $T : V_pM \to T_pM$ as $[\gamma] \to \gamma'(0)$. (Why is this well-defined?) This map is a bijection (why?) Thus this canonical map can be used to define the vector space structure such that it is a linear isomorphism.

Recall that if $f : \mathbb{R}^n \to \mathbb{R}^k$ $(k < n)$ is a smooth map such that Df_a has full rank=k, that is, it is surjective whenever $f(a) = 0$, then $f^{-1}(0)$ "can be made into" a smooth manifold (HW 3). By the way, why $k < n$? What about the image of a smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$ $(m > n)$, i.e., Df is $1-1$? Even one where Df has full rank everywhere? One can find a counterexample where f is *also* $1-1$ in addition to Df being $1-1$ everywhere! ($f : (-\pi, \pi) \to \mathbb{R}^2$ given by $f(t) = (\sin(2t), \sin(t))$). So if $f : M \to N$ (manifolds without boundary) is a smooth map $(n < m)$, $q \in N$, such that $f_* : T_pM \to T_{f(p)=q}N$ is surjective whenever $f(p) = q$, then can $f^{-1}(q)$ be made into a smooth manifold? Likewise, what about the other case?

2 Immersions, submersions, and embeddings

Definitions: Let M, N be smooth manifolds (with or without boundary) and $F : M \to N$ be a smooth map. The rank of F at p is defined to be the rank of $(F_*)_p : T_pM \to T_{F(p)}N$ (which is the same as the rank of $[DF]_p$ in coordinate charts). If F has the *same* rank at every point, then it is said to have constant rank. If $(F_*)_p$ has full rank, then F is said to have full rank at p. If $(F_*)_p$ is surjective for *all* $p \in M$, then F is called a submersion. It is $1-1$ for all $p \in M$, then F is said to be an immersion.
Proposition: If $(F_*)_p$ is surjective, then p has a neighbourhood U such that $F : U \to N$ is a submersion. Likewise for injectivity at p.
Proof: Indeed, choosing coordinates, the smooth matrix-valued function $[DF]$ has full rank at p iff a minor is non-zero. That minor will continue to be non-zero in a neighbourhood. $\square$

Examples and non-examples:

- $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is *not* of constant rank. It is an immersion (and a submersion) at $x = 1$ for instance.
- $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x$ is a submersion. Likewise for projections from products of manifolds.
- $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by $f(x, y) = (x, y, 0)$ is an immersion. Likewise for inclusions into products of manifolds.
- Let $\gamma : J \to M$ be a smooth map. Then γ is an immersion iff $\gamma'(t) \neq 0$ for all $t \in J$.
- A circle rotated about an axis can be thought of as an immersion of $\mathbb{R}^2$ into $\mathbb{R}^3$.
- A $1-1$ immersion need *not* be a homeomorphism to its image.

Our figure-8 1-1 immersion was not a manifold. The key problem is that the map was not a homeomorphism of $(-\pi, \pi)$ to its image (which was a manifold).
Definition: Let M, N be smooth manifolds (with or without boundary). A smooth map $F : M \to N$ is called a smooth embedding if it is a $1-1$ immersion *and* $F : M \to F(M)$ is a *homeomorphism*.
Examples:

- Let $U \subset M$ be an open subset. Then the inclusion map $i : U \subset M$ is a smooth embedding: Indeed, it is a smooth $1-1$ immersion. Its topology is induced from M and hence of course homeomorphic to its image.

- The inclusion map $M_i \to M_1 \times M_2 \ldots M_k$ given by $f(q) = (p_1, p_2, \ldots, p_{i-1}, q, p_{i+1}, \ldots)$ is a smooth embedding. In particular, the inclusion of $\mathbb{R}^n$ into $\mathbb{R}^{n+k}$ is a smooth embedding.

- It turns out that (HW) a torus treated as a surface of revolution gives a smooth embedding into $\mathbb{R}^3$.

Proposition: If $F : M \to N$ is a $1-1$ immersion, then F is a smooth embedding if either F is an open or a closed map or if M is compact.
Proof: If F is open or closed, it is a homeomorphism to its image. If M is compact, then since N is Hausdorff, F is closed and hence a homeomorphism to its image.
S^n is a manifold in its own right. It is also a subset of another manifold $\mathbb{R}^{n+1}$. Are the smooth structures "compatible"?
Definitions: Let M be a manifold (with or without boundary) and let $S \subset M$ be a subset that carries a smooth manifold (without boundary) structure. If the inclusion map $i : S \to M$ is a smooth embedding, then S is said to be an embedded submanifold (or simply a submanifold) of the ambient manifold M. If i is merely a $1-1$ immersion, then S is said to be an *immersed* submanifold of M. The *codimension* of a submanifold S is $dim(M) - dim(S)$.

Examples:

- The figure-8 is an immersed but not embedded submanifold of $\mathbb{R}^2$.

- The linear subspace $\mathbb{R}^n$ is an embedded submanifold of $\mathbb{R}^m$ when $m > n$.

- An open subset $U \subset M$ is an embedded submanifold.

- S^n is an embedded submanifold of $\mathbb{R}^{n+1}$.

- Every "slice" of $M \times N$ is an embedded submanifold.

- Graphs are embedded submanifolds.

- It turns out (HW) that the boundary of a manifold with boundary is an embedded submanifold (without boundary) of codimension 1.

Is every manifold secretly a submanifold of $\mathbb{R}^N$?
Whitney's embedding theorem: Every smooth n-manifold with or without boundary admits a smooth embedding into $\mathbb{R}^{2n+1}$.
This theorem is akin to Cauchy's theorem of group theory. The proof is tricky. We shall prove a weak version (Not $N = 2n + 1$) of it only for compact manifolds without boundary, later.

3 Function theorems on manifolds

How can we come up with examples of embedded submanifolds? (HW 3) suggests that having inverse/implicit function type theorems on manifolds can help. From now onwards, we will focus mainly on manifolds without boundary. Towards the end of this course, we will again come back to manifolds-with-boundary (for Stokes' theorem).

Inverse function theorem on manifolds: Let M, N be smooth manifolds without boundary and $F : M \to N$ be a smooth map. If $(F_*)_p : T_pM \to T_{F(p)}N$ is invertible, then F is a local diffeomorphism, i.e., there exist connected neighbourhoods U, V of $p, F(p)$ such that $F : U \to V$ is a diffeomorphism.

Proof: Choose coordinate charts $(\tilde{U} \subset M, x)$ and $(\tilde{V} \subset N, y)$ centred at $p, F(p)$. In these charts (abusing notation), $(F_*)_p$ is $[DF]_p$ which is assumed to be invertible. Thus, by the usual IFT, F is a local diffeomorphism. $\square$

Constant Rank Theorem: Suppose M, N are manifolds (without boundary) and $F : M \to N$ is a smooth map with constant rank r. For every $p \in M, F(p) \in N$, there exist charts so that $\hat{F}(x^1, \ldots, x^m) = (x^1, \ldots, x^r, 0, 0, \ldots)$.

Proof: Choose some arbitrary charts centred at $p, F(p)$. Now the problem is a local one, i.e., if $F : U \subset \mathbb{R}^m \to \mathbb{R}^n$ is a smooth map with constant rank r and $F(0) = 0$, then we need to prove that there exist local diffeos $\phi : V \subset U \to \phi(V) \subset \mathbb{R}^m$ and $\psi : W \subset \mathbb{R}^n \to \psi(W)$ such that $\hat{F} = \psi \circ F \circ \phi^{-1}$ has the desired form. We shall abuse notation and denote $\hat{F}$ by F as always.

Using appropriate linear transformations, we can ensure that $DF(0)$ is of the form $\begin{bmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{bmatrix}$ (why?) We need to use the IFT or ImFT to choose charts (by a nonlinear transformation) so that this behaviour of $DF(0)$ translates into the same kind of behaviour for F itself. Consider the map $G : U \to \mathbb{R}^m$ given by $G(x) = (F^1, \ldots, F^r, x^{r+1}, x^{r+2}, \ldots)$. Now G is smooth and $DG(0) = I$. Thus by IFT, G is a local diffeo. Choose $\phi = G$ itself. Then $F \circ \phi^{-1}(y) = F \circ G^{-1}(y) = (y^1, \ldots, y^r, F^{r+1}(x(y)), \ldots)$.

Now we use the constant rank hypothesis to conclude that $F \circ \phi^{-1}(y)$ does not depend on $y^{r+1}, \ldots$ (why?). Thus $F \circ \phi^{-1}(y) = (y, S(y))$ for some smooth S. We need to change coordinates in the target to make sure that S becomes zero. Define $\psi(u, v) = (u, v - S(u))$ so that the second half is 0 iff $v = S(u)$. Thus if ψ is a valid local change of coordinates, then $\hat{F}(y) = (y, 0)$. ψ has an explicit inverse and is a diffeo (why?) $\square$

Slice charts: We want to model embedded submanifolds by means of the standard inclusion $\mathbb{R}^m \to \mathbb{R}^n$. (In particular, we want to say embedded submanifolds are locally graphs of smooth functions.) This means that we want to choose nice charts to make this happen. More generally, we say that if $U \subset \mathbb{R}^n$ is open, then a k-slice of U is $x^{k+1} = c^{k+1}, x^{k+2} = c^{k+2}, \ldots$, i.e., set all except for k coordinates to constants. Alternatively, simply consider the graph of a constant function. If M is a manifold (without boundary) and $S \subset M$, then S is said to be a local k-slice near p if there exists a chart (ϕ, U) near p so that $S \cap U$ is a k-slice in this chart. (By the way, we can always make sure that the constants are 0 by subtraction.)

Theorem (Slice charts exist for embedded submanifolds): If $S \subset M$ is a k-dimensional embedded submanifold, then S is a local k-slice for all $p \in S$. Conversely, if $S \subset M$ is a subset that is a local k-slice for all $p \in S$, then with the subspace topology S is a topological k-fold. Moreover, it has a smooth structure making it into a k-dimensional embedded submanifold. (As we shall see later, this is the *unique-upto-diffeo* smooth structure on S making it into a submanifold.)