NOTES FOR 10 NOV (FRIDAY)

1. Recap

(1) We proved the chain homotopy formula.
(2) Defined De Rham cohomology (and cohomology with compact support). Defined the pullback of cohomology classes.
(3) Saw some examples of cohomology classes. Just started proving a theorem.

2. De Rham cohomology

Theorem 2.1. *If M is a connected orientable m-manifold, then the map $T[\omega] = \displaystyle\int_M \omega$ gives an isomorphism $H_c^m(M) \simeq \mathbb{R}$.*

Proof. (1) True on $\mathbb{R}$ and S^1 : If ω is a 1-form with compact support on $\mathbb{R}$ such that $\int_{\mathbb{R}} \omega = 0$, then define a function $f(x) = \displaystyle\int_{-\infty}^{x} \omega$. Note that $df = \omega$ by the FTC. f has compact support because $f(-a) = 0$ for a large positive a and since $\int_{\mathbb{R}} \omega = \int_{-a}^{b} \omega$ for all $b \geq a$, we see that $f(x) = 0$ for $x > a$. Hence the theorem holds for $\mathbb{R}$. Likewise, if $\omega = g(\theta)d\theta$ where $g(0) = g(2\pi)$, then $f(\theta) = \int_0^{\theta} g(\theta)d\theta$ satisfies $df = \omega$ and $f(0) = 0 = f(2\pi)$. Thus the theorem holds for the circle too.

(2) True on $\mathbb{R}^m$ assuming that it is true on S^{m-1} : Let $\omega = f dx^1 \wedge \ldots dx^m$ be a top form with compact support in $\mathbb{R}^m$ such that $\int \omega = 0$. Assume wlog that ω is supported in the unit ball. We know that $\eta = \displaystyle\sum_{i=1}^{m} (-1)^{i-1} \int_0^1 x^i t^{m-1} f(tx) dt dx^1 \ldots \hat{dx}^i \wedge dx^m$ satisfies $d\eta = \omega$ (by an exercise). We will modify η by another exact form so that the resulting form has compact support.
Since we want to use that $f(u\vec{n})$ vanishes for large u, define $u = |x|t$ to get

$$\eta(x) = \sum_{i=1}^{m} (-1)^{i-1} \int_0^{|x|} u^{m-1} f(u\frac{x}{|x|}) du \frac{1}{|x|^m} dx^1 \ldots \hat{dx}^i \wedge dx^m \tag{2.1}$$

If $|x| > 1$, then the integral need be evaluated only till 1. In other words, for large values of $|x|$, η depends mainly on its values on the unit sphere. We will prove that in fact, $\eta = d\gamma$ for some $m-2$ smooth form γ outside the unit ball. Let h be a smooth function equal to 1 outside the ball of radius 2 and equal to 0 on the unit ball. Then $h\gamma$ is well-defined on all of $\mathbb{R}^m$ and satisfies, $\omega = d(\eta - d(h\gamma))$ and moreover, $\eta = d(h\gamma)$ outside a compact set.

Indeed, outside the unit ball,

$$\eta(x) = \sum_{i=1}^{m} (-1)^{i-1} \int_0^{1} u^{m-1} f(u\frac{x}{|x|}) du \frac{1}{|x|^m} dx^1 \ldots \hat{dx}^i \wedge dx^m \tag{2.2}$$

Now define $g : S^{m-1} \to \mathbb{R}$ by $g(x) = \int_0^1 u^{m-1} f(ux) du$. Define $r(x) : \mathbb{R}^m - 0 \to S^{m-1}$ as $r(x) = \frac{x}{|x|}$. It is a retraction to the unit circle. Suppose σ' is the natural orientation form on

S^{m-1} defined as $\sigma'_x(v_1,\ldots,v_{m-1}) = \det(\vec{x}, v_1, \ldots, v_{m-1})$.

We claim that

$$\frac{1}{|x|^m} dx^1 \ldots \hat{dx}^i \wedge dx^m = r^*\sigma'.$$

Firstly, σ' is the restriction to S^{m-1} of the form $\sigma = \sum(-1)^{i-1}x^i dx^1 \ldots \hat{dx}^i \ldots dx^m$. Indeed, $\sigma(v_1,\ldots,v_{m-1}) = \sum(-1)^{i-1}x^i dx^1 \ldots \hat{dx}^i \ldots dx^m(v_1,\ldots,v_{m-1}) = \det(\vec{x}, v_1, \ldots, v_{m-1})$ by evaluating along the first row.

Secondly, we compute $r^*(\sigma)$:

$$\begin{aligned} r^*\sigma &= \sum(-1)^{i-1} r^*(x^i) dr^*(x^1)\ldots \\ &= \sum(-1)^{i-1}\frac{x^i}{|x|} d\left(\frac{x^i}{|x|}\right)\ldots \end{aligned} \tag{2.3}$$

which after a tedious calculation (using $d(f/g) = df/g - f dg/g^2$) shows the claim.

Therefore, outside the unit ball, $\eta(x) = g(x) r^*\sigma' = r^*(g\sigma')$. Also, we claim that $\int_{S^{m-1}} g\sigma' = \int_B f dx^1 \ldots dx^m = \int \omega_{\mathbb{R}^m} = 0$. Indeed, by Stokes, $\int_{S^{m-1}} g\sigma' = \int_B d(g\sigma) = \int_B \omega = 0$.

Now, by the induction assumption, the result holds for $m-1$ dimensional manifolds. In particular, it holds for S^{m-1}. Therefore, $g\sigma' = d\gamma$ for some form γ on S^{m-1}. Therefore, $\eta = r^* d\gamma$.

$\square$