# NOTES FOR 11 AUG (FRIDAY)

## 1. Recap

(1) Defined manifolds-with-boundary and gave examples. (Note that if you take the product of two manifolds with boundary, then you get one which has corners, i.e., the usual construction of charts $U \times V, (\Phi_U, \Phi_V)$ does not give you a manifold-with-boundary because at the corners, this object does not look like an open set of the upper half space.)
(2) Defined the notion of a smooth map between manifolds (of possibly different dimensions). Using bump functions, we proved a smooth version of Urysohn's lemma (i.e. there are bump functions on manifolds).
(3) Defined the notion of partial derivatives of smooth maps $f : N \to \mathbb{R}$. Unfortunately, it depended on the choice of coordinate chart. Today we will see what happens when you change the coordinate chart.

## 2. Maps to manifolds and all that

Here is the statement that the Chain rule holds for partial derivatives of functions on manifolds. (As Spivak says, it is just the usual chain rule if you keep your cool.)

**Lemma 2.1.** *If $(x = \Phi_U, U)$ and $(y = \Phi_V, V)$ are two coordinate systems on $N$, and $f : N \to \mathbb{R}$ is differentiable, then on $U \cap V$ we have the following "transformation rule" for the partial derivatives of $f$ :*

$$\frac{\partial f}{\partial y^i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} \frac{\partial x^j}{\partial y^i} \tag{2.1}$$

*Proof.* Indeed,

$$\begin{aligned} \frac{\partial f}{\partial y^i} &= \frac{\partial f \circ \Phi_V^{-1}}{\partial y^i} = \frac{\partial f \circ x^{-1} \circ x \circ y^{-1}}{\partial y^i} \\ &= \sum_{j=1}^{n} \frac{\partial f \circ x^{-1}}{\partial x^j} \frac{\partial (x \circ y^{-1})^j}{\partial y^i} = \sum_j \frac{\partial f}{\partial x^j} \frac{\partial x^j}{\partial y^i} \end{aligned} \tag{2.2}$$

□

**Remark 2.2.** There is nothing fancy about what happened so far. It is as straightforward as saying, in $\mathbb{R}^2$, $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$.

**Remark 2.3.** Don't get bogged down by the notation. Later on, instead of the cumbersome notation $\tilde{\Phi}_{\tilde{U}} \circ f \circ \Phi_U^{-1} : \Phi_U(U) \subset \mathbb{R}^m \to \tilde{\Phi}_{\tilde{U}} \subset \mathbb{R}^n$, we will abuse notation and forget about the $\Phi$. By default we will (from now onwards) identify $U$ with $\Phi_U(U) \subset \mathbb{R}^m$ and write $\Phi_U^{-1}(p) = (x^1(p), x^2(p), \ldots, x^m(p))$ (the superscripts are there for a reason), and $\tilde{\Phi}_{\tilde{U}} \circ f(p) = (f^1(p), f^2(p), \ldots, f^n(p))$. So the partial derivative $\frac{\partial f}{\partial x^i}$ is indeed what you think it is.

**Remark 2.4.** The point of using superscripts for coordinates is the Einstein summation convention : If two indices $j$ appear "above" and "below", then they are summed over by default. In other words, $\displaystyle\sum_{j=1}^{n}\frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial y^i}$ may be simply written as $\frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial y^i}$ (omitting the $\sum$ symbol with the understanding that repeated indices are summed over.

**Remark 2.5.** Already one can see the theme "study quantities that behave well under coordinate changes". Note that the partial derivative $\frac{\partial f}{\partial x^i}$ does not make sense as a function from $N$ to $\mathbb{R}$ but instead it is a collection of locally defined functions that "transform" according to the chain rule above. Later on, we will interpret this "derivative" as a function not from $N$ to $\mathbb{R}^n$, but instead to another, appropriately defined manifold (called by a fancy name - the cotangent bundle $T^*N$).

It is useful to think of $l = \frac{\partial}{\partial y^i}$ as an operator and the chain rule as an operator equation $\frac{\partial}{\partial y^i} = \frac{\partial x^j}{\partial y^i}\frac{\partial}{\partial x^j}$. In this regard, we have a property that we will return to later. (The proof is trivial.)

**Lemma 2.6.** *For any two differentiable functions $f, g : M \to \mathbb{R}$, and any coordinate system $(x, U)$ with $p \in U$, the operator $l = \frac{\partial}{\partial x^i}$ satisfies $l(fg)(p) = f(p)l(g)(p) + l(f)g(p)$.*

Note that the matrix $\frac{\partial x^i}{\partial y^j}$ is non-singular (because we demanded for a manifold, the transition functions $x \circ y^{-1}$ be diffeomorphisms. Therefore, they are differentiatible and so are there inverses. The inverse of the matrix $\frac{\partial x^i}{\partial y^j}$ is the matrix $\frac{\partial y^i}{\partial x^j}$. Thus, if $f : M^m \to N^n$ is a smooth map, then choosing coordinates $p \in (x, U)$ in $M$ and $f(p) \in (\tilde{x}, V)$ in $N$, the rank of the $n \times m$ matrix $Df^i_{\ j} = \frac{\partial f^i}{\partial x^j}$ is *independent of the coordinate charts* $(x, U)$, $(\tilde{x}, V)$. Thus the following definition makes sense :

**Definition 2.7.** Suppose $f : M^m \to N^n$ is a $C^1$ map. Assume that $p \in (x, U)$ and $f(p) \in (\tilde{x}, \tilde{U})$ are any two coordinate charts on $M$ and $N$ respectively. Then $f$ is said to be

(1) An *immersion* at $p \in M$ if the matrix $Df^i_{\ j}(p)$ has no kernel.
(2) A *submersion* at $p \in M$ if the matrix $Df^i_{\ j}(p)$ is onto.

In other words, it is an immersion if $m \leq n$ and the derivative matrix has full rank, and a submersion if $m \geq n$ and the derivative matrix has full rank. If $f$ is a **submersion** at $p$, $p$ is said to be a **regular point**. If not, then $f(p)$ is said to be a **critical value** and $p$ a **critical point**. Points in $N$ other than critical values are called **regular values**, i.e., $a \in N$ is a regular value if and only if every point $p \in f^{-1}(a)$ is a regular point. So in particular, if $a$ is not even in the image of $f$ (i.e. $a$ is not a "value" of $f$), even then it is by definition counted as a regular value.

What is the point of this definition ? We will discuss that in a moment. Firstly, here are some examples and counterexamples of immersions :

(1) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is 1-1 but NOT an immersion.
(2) Likewise, $g(x) = e^{-x^2}$ when $x > 0$, $g(0) = 0$, $g(x) = -e^{-x^2}$ when $x < 0$ is 1-1 but NOT an immersion. A stranger example is $\vec{h}(x) = (g(x), |g(x)|)$ whose graph has a corner but the curve itself is differentiable by slowing down to 0 speed at the corner.
(3) $\vec{c}(t) = (\cos(t), \sin(t))$ from $\mathbb{R} \to \mathbb{R}^2$ is an immersion but is not 1-1. In fact, one can come up with "knot" shaped curves that are immersions but intersect themselves.
(4) The image of an immersion $f : M_1 \to M$ can even be a dense subset of $M$. Indeed, take an irrational winding of the torus. It is $1 - 1$ and an immersion.

(5) If you think that the images of 1-1 immersions of $\mathbb{R}$ are homeomorphic to $\mathbb{R}$, think again. Look at the standard picture on page 48 in Spivak.

Here are some examples of regular and critical points/values :

(1) $f : \mathbb{R} \to \mathbb{R}$ has a critical point at $x$ if and only if $f^{'}(x) = 0$. So all points of $\mathbb{R}$ can be critical points only for constant functions.
(2) If we take $f : \mathbb{R} \to \mathbb{R}^2$, then all points of $\mathbb{R}$ are critical points of $f$ simply because a linear map from a 1 dimensional vector space cannot be onto a 2 dimensional one.
(3) $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x, 0)$ has all points in the domain as critical points. However, the critical values are only points on the x-axis, a "small set", so to say. In fact, this is a prototypical example of this phenomenon (called Sard's theorem). We will return to this later.

Our aim now is to define the notion of a "submanifold" and produce examples of submanifolds (which is where we will need the words "regular value", "critical value", etc). A submanifold $S$ of a manifold $M$ should be morally speaking, a subset $S \subset M$ which is a manifold in its own right, and such that the manifold structure of $S$ is in some sense "compatible" with that of $M$. There are two possible ways to make this rigorous :

**Definition 2.8.** Suppose $S \subset M$ is a manifold of dimension $s$. Let $i : S \to M$ be the inclusion map. $S$ is said to be

(1) an *immersed submanifold* if $i$ is a smooth immersion.
(2) an *embedded submanifold* if $i$ is a smooth immersion which is also homeomorphic to its image.

If an embedded submanifold is a closed subset of $M$, then it is called a *closed submanifold* (although some (annoying) people refer to compact manifolds without boundary as closed submanifolds).

As we saw, an immersed submanifold can be a strange object (not being an embedding or being dense, etc). In fact, if $P$ is a smooth manifold and $f : P \to M$ is a smooth function such that $f(P) \subset M_1$ where $M_1$ is an immersed submanifold of $M$, then it $f$ might not even be *a continuous function as a map into $M_1$!* Actually, this is the only thing that can go wrong. But we will see this phenomenon a little later.

In any case, how does one produce examples of embedded submanifolds ? Even those of $\mathbb{R}^n$ ? Recall the implicit function theorem (IFT) from multivariable calculus :

**Theorem 2.9.** *If $\vec{F}(\vec{x}, \vec{y}) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is a $C^1$ function such that at a point $(p, q)$, the matrix $\frac{\partial F^i}{\partial y^j}$ is invertible, then locally around $(p, q)$, we can solve for $\vec{y}$ as $\vec{y} = \vec{g}(\vec{x})$ for some $C^1$ function $\vec{g}$.*

**Remark 2.10.** This is a corollary of the inverse function theorem (which is the same as above with $m = 0$, i.e., if the derivative is invertible at a point, the function is a local diffeomorphism).