NOTES FOR 11 OCT (WEDNESDAY)

1. Lie algebras (cont'd..)

Theorem 1.1. *Let G be a Lie group and $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra. Then there is a unique connected Lie subgroup H of G whose Lie algebra is $\mathfrak{h}$.*

Very brief sketch of the proof : Just from the tangent space at e, we need to produce a full Lie subgroup ! How does one produce a manifold from such little information ? Well, one produce a bunch of subspaces $\Delta_a = (L_a)_* \mathfrak{h}$. Next one shows that this is a distribution (indeed just take a basis at e and left-translate). Because $\mathfrak{h}$ is a Lie subalgebra, we see that this distribution is integrable. Now take the maximal integral submanifold containing the identity (which exists by Frobenius). Prove that this is a Lie subgroup. Uniqueness is not very hard.

A difficult theorem of Ado shows that every Lie algebra is isomorphic to a subalgebra of $GL(N, \mathbb{R})$.

Suppose $\phi : G \to H$ is a Lie group homomorphism. Then firstly $\phi_* : T_e G \to T_e H$ is a vector space map. Secondly, note that $\phi \circ L_a = L_{\phi(a)} \circ \phi$ by definition. Let $X_e \in T_e G$, $Y_e \in T_e G$ be two Lie algebra elements and $X(a) = L_{a*} X_e, Y(a) = L_{a*} Y_e$ be the corresponding unique left-invariant extensions to G. Let $\tilde{X}(b) = L_{b*} \phi_* X_e, \tilde{Y}(b) = L_{b*} \phi_* X_e$ be the left-invariant extensions on H. Now $\tilde{X}(\phi(a)) = \phi_* L_{a*} X_e = \phi_*(X(a))$ and likewise for Y. Actually, the calculation of the lemma above can be generalised (and simplified) to the following lemma :

Lemma 1.2. *Suppose $f : M \to N$ is a smooth map. Assume that $\tilde{X}, \tilde{Y}$ are smooth vector fields on N, and X, Y are smooth vector fields on M such that $\tilde{X}(f(a)) = f_*(X(a))$ and likewise for Y. Then $[\tilde{X}, \tilde{Y}]_{f(a)} = f_*[X, Y]_a$.*

Proof. Instead of using coordinates, this time we will do it by definition. Let $g : f(a) \in U \subset N \to \mathbb{R}$ be a smooth function. Then, firstly note that

$$\begin{aligned} \tilde{Y}_{f(p)}(g) &= f_* Y_p(g) = Y_p(g \circ f) \\ \Rightarrow (\tilde{Y} g) \circ f &= Y(g \circ f) \end{aligned} \tag{1.1}$$

and likewise for X. Now

$$\begin{aligned} ([\tilde{X}, \tilde{Y}] g) \circ f &= (\tilde{X} \tilde{Y} g) \circ f - (\tilde{Y} \tilde{X} g) \circ f = X(\tilde{Y} g \circ f) - Y(\tilde{X} g \circ f) \\ &= X(Y(g \circ f)) - Y(X(g \circ f)) = [X, Y](g \circ f) \end{aligned} \tag{1.2}$$

□

Therefore, given a Lie group homomorphism $\phi : G \to H$, $\phi_* : \mathfrak{g} \to \mathfrak{h}$ induces a Lie algebra homomorphism.

Here are a couple of examples :

1. If $G = \mathbb{R}$, $H = \mathbb{R}$ with addition as the operation, then $\phi(s + t) = \phi(s) + \phi(t)$. The only smooth functions of this sort are $\phi(t) = ct$. Therefore, $\phi_* : \mathbb{R} \to \mathbb{R}$ is multiplication by c.
2. If $G = \mathbb{R}$, $H = S^1 \subset \mathbb{C}$, then $\phi(s + t) = \phi(s)\phi(t)$. Therefore $\phi'(t) = \phi(t)\phi'(0)$ and hence $\phi(t) = e^{\sqrt{-1}(ct+k)}$. Thus ϕ_* is multiplication by $\sqrt{-1}c$.

(3) If $G = S^1$ and $H = \mathbb{R}$, and ϕ is smooth (actually even continuous should do), then since the image is compact and $\phi(e^{\sqrt{-1}nx}) = n\phi(e^{\sqrt{-1}x})$ (and suppose x is irrational), then the image is 0. Thus there could be Lie algebra homomorphisms $\mathfrak{g} \to \mathfrak{h}$ that may not come from smooth Lie group homomorphisms.

However, we have a local result.

Theorem 1.3. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$, $\mathfrak{h}$. Let $\Phi : \mathfrak{g} \to \mathfrak{h}$ be a homomorphism. Then there is a neighbourhood of e and a smooth map ϕ such that $\phi(ab) = \phi(a)\phi(b)$ and $\phi_* X_e = \Phi(X_e)$. Moreover, if there are two smooth homomorphisms inducing the same Lie algebra homomorphism and G is connected, then they agree.*

We will omit the proof of this result. Here are some interesting corollaries though.

Corollary 1.4. *If two Lie groups G and H have isomorphic Lie algebras, then they are locally isomorphic.*

Proof. The Lie algebra isomorphism $\Phi : \mathfrak{g} \to \mathfrak{h}$ is locally induced around the identity by $\phi : U \subset G \to H$ according to the previous theorem. By the inverse function theorem, locally ϕ is a diffeomorphism and hence an isomorphism around the identity.
Suppose $a \in G$, then $\phi \circ L_{a^{-1}}$ is the local isomorphism near a. □

Corollary 1.5. *A connected Lie group G with an abelian Lie algebra is abelian.*

Proof. Any abelian Lie algebra is isomorphic to $\mathbb{R}^n$ and hence G is locally isomorphic to $\mathbb{R}^n$. So G is locally abelian. Every neighbourhood of e generates G. This is a problem in Spivak (will be given as HW). Thus we are done. □

Corollary 1.6. *For every $X_e \in T_eG$, there is a unique smooth homomorphism $\phi : \mathbb{R} \to G$ such that $\phi_*(\frac{\partial}{\partial t}_{t=0}) = X_e$.*

Spivak gave two proofs. They are both instructive.

Proof. Define a Lie algebra homomorphism $\Phi : \mathbb{R} \to \mathfrak{g}$ given by $\Phi(a) = aX_e$. This induces a Lie group homomorphism (by the above theorem) locally $\phi : (-\epsilon, \epsilon) \to G$ such that $\phi_*(\frac{\partial}{\partial t}_{t=0}) = X_e$. Now given a $t \in \mathbb{R}$, by composition, one can extend ϕ to all of $\mathbb{R}$. More precisely, suppose $t = k\frac{\epsilon}{2} + r$ where $k \in \mathbb{Z}$ and $|r| < \frac{\epsilon}{2}$. Then

$$\phi(t) = \phi(\frac{\epsilon}{2})\phi(\frac{\epsilon}{2})\ldots\phi(\frac{\epsilon}{2})\phi(r) \tag{1.3}$$

if $k \geq 0$ and likewise

$$\phi(t) = \phi(-\frac{\epsilon}{2})\phi(-\frac{\epsilon}{2})\ldots\phi(-\frac{\epsilon}{2})\phi(r) \tag{1.4}$$

otherwise. Uniqueness also follows from the above theorem. □

Proof. If $\tilde{X}(a) = L_{a*}X_e$ is the unique left-invariant extension, then consider the flow of this vector field through e. Indeed, if $\phi(t)$ is the integral curve (such that $\phi(0) = e$, $\phi'(0) = X_e$), then it can be extended to all of $\mathbb{R}$ by means of the method of the first proof. Now we know that $\phi(s)\phi(t)$ is the integral curve passing through $\phi(s)$ at $t = 0$. The same is true for $\phi(t+s)$. By uniqueness, $\phi(t+s) = \phi(t)\phi(s)$. Thus we have $\phi : \mathbb{R} \to G$ a homomorphism.

As for uniqueness, suppose $\phi : \mathbb{R} \to G$ is a homomorphism such that $\phi_*(\frac{\partial}{\partial t}_{t=0}) = X_e$, then $\phi(t)$ is an integral curve through e of $\tilde{X}(a) = L_{a*}X_e$ and is hence unique. Indeed,

$$\phi_*(\frac{\partial}{\partial t})(f) = \frac{d(f \circ \phi)}{dt} = \lim_{h \to 0} \frac{f(\phi(t)\phi(h)) - f(\phi(t))}{h}$$
$$= \frac{d(f \circ L_{\phi(t)} \circ \phi(u))}{du}|_{u=0} = L_{\phi(t)*}X(f) = \tilde{X}_{\phi(t)}(f). \tag{1.5}$$

□

A homomorphism $\phi : \mathbb{R} \to G$ is called a 1-parameter subgroup of G. There is a unique 1-parameter subgroup with a given tangent vector at the identity. For $\mathbb{R}$, we know all of them are of the form $\phi(t) = ct$. For the multiplicative group $\mathbb{R} - 0$, $\phi(s+t) = \phi(s)\phi(t)$ and hence $\phi(t) = e^{ct}$. Likewise, if we want the one-parameter subgroups of $GL(n, \mathbb{R})$, they satisfy $\phi(s+t) = \phi(s)\phi(t)$ and hence $\frac{d\phi}{dt} = \phi'(0)\phi(t)$. The solution of this system is $\phi(t) = e^{Ct}$ where e^A is defined as $e^A = I + A + \frac{A^2}{2!} + \ldots$. Indeed, firstly e^A is well-defined because if $||A||$ is the operator norm of A, then $||A + B|| \leq ||A|| + ||B||$ and $||AB|| \leq ||A||||B||$. Thus $||\frac{A^n}{n!}|| \leq \frac{||A||^n}{n!}$. Therefore the series converges in this norm (and hence in every norm). Moreover, if $AB = BA$, then $e^{A+B} = e^A e^B$. (Will be given as HW again.) Therefore,

$$\phi'(t) = \lim_{h \to 0} \frac{e^{C(t+h)} - e^{Ct}}{h} = e^{Ct} \lim_{h \to 0} \frac{e^{Ch)} - I}{h}$$
$$= e^{Ct} \lim_{h \to 0} \frac{Ch + \frac{C^2h^2}{2!} + \ldots}{h} = Ce^{Ct} \lim_{h \to 0} (I + \frac{Ch}{2!} + \ldots) = Ce^{Ct} \tag{1.6}$$

Motivated by this, for any Lie group G we define the exponential map as $\exp : \mathfrak{g} \to G$: If $X_e \in \mathfrak{g}$, extend X_e to $X(a) = L_{a*}X_e$ and let $\phi(t)$ be the flow of this field through the identity. Then $\exp(X_e) = \phi(1)$. By the property of flows we see that $\exp(-X_e) = (\exp(X_e))^{-1}$ and $\exp((s+t)X_e) = \exp(sX_e)\exp(tX_e)$. Note that the exponential map on $GL(n)$ is simply the matrix exponential.