NOTES FOR 11 SEPT (MONDAY)

1. Recap

(1) Proved that if $X(p) \neq 0$ then X is locally a coordinate vector field.
(2) Defined the Lie derivative $L_X Y$, $L_X \omega$, proved it exists, is bilinear, satisfies the product rule, and then we produced a formula to compute it.

2. Vector fields, Tangent bundle, Cotangent bundle, etc

Lemma 2.1. $L_X Y(f) = X(Y(f)) - Y(X(f)) = [X, Y](f)$

Proof. $X(Y(f)) = X(Y^i \frac{\partial f}{\partial x^i}) = X^j \frac{\partial Y^i}{\partial x^j} \frac{\partial f}{\partial x^i} + X^j Y^i \frac{\partial^2 f}{\partial x^i \partial x^j}$. Therefore $[X, Y] = L_X Y$. ☐

3. Lie bracket, Frobenius theorem

Note that the Lie derivative satisfies the following properties (which are apparent from $L_X Y = [X, Y]$.

(1) $L_X(aY + bZ) = aL_X Y + bL_Z Y$.
(2) $L_{aX_1 + bX_2} Y = aL_{X_1} Y + bL_{X_2} Y$
(3) $L_X Y = -L_Y X$ and hence $L_X X = 0$.
(4) $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$ (the Jacobi identity).

Any vector space g endowed with a map (called the Lie bracket) $[,] : g \times g \to g$ is called a Lie algebra. The space of vector fields is a Lie algebra with bracket $L_X Y = [X, Y]$.

The next order of business is to provide a geometric interpretation for the "Lie bracket" $[X, Y]$. Prior to that we prove the following lemma

Lemma 3.1. *Let X generate ϕ_t and Y generate ψ_t. Then $[X, Y] = 0$ if and only if $\psi_s \circ \phi_t = \phi_t \circ \psi_s$ $\forall t, s$.*

Proof. In local coordinates (x, U), suppose $V \subset U$ is an open set and $|s|, |t| < \epsilon$ where ϵ is small enough so that $\phi_t(V) \subset U$, $\psi_s(V) \subset U$, we see that

$$\frac{\partial^2}{\partial s \partial t}(\psi_s \circ \phi_t - \phi_t \circ \psi_s) = \frac{\partial^2}{\partial s \partial t}(\psi(s, \phi(t, p)) - \phi(t, \psi(s, p)))$$

$$= \frac{\partial}{\partial t} Y(\psi(s, \phi(t, p))) - \frac{\partial}{\partial s} X(\phi(t, \psi(s, p))) = \frac{\partial Y^i}{\partial x^j} X^j(\psi_s \circ \phi_t(p)) e_i - \frac{\partial X^i}{\partial x^j} Y^j(\phi_s \circ \psi_t(p)) e_i \tag{3.1}$$

Of course if $\psi_s \circ \phi_t = \phi_t \circ \psi_s$, then $[X, Y] = 0$.
For the converse, firstly note that if α is a diffeomorphism, then $\alpha_* X$ is a vector field whenver X is. Moreover, $\alpha_* X$ generates $\alpha \circ \phi_t \circ \alpha^{-1}$. Indeed,

$$(\alpha_* X)_q(f) = \alpha_* X_{\alpha^{-1}(q)} f = X_{\alpha^{-1}(q)}(f \circ \alpha) = \lim_{h \to 0} \frac{(f \circ \alpha) \circ \phi_h(\alpha^{-1}(q)) - f(q)}{h} \tag{3.2}$$

This implies that $\alpha_* X = X$ if and only if $\alpha \circ \phi_t = \phi_t \circ \alpha$. (By uniqueness of flows.) Thus, we simply need to prove that $\phi_{t*} Y = Y$. Now let $c(t) = (\phi_{t*} Y)(p)$. This is a map into $T_p M$. Its derivative is $c'(t) = \lim_{h \to 0} \frac{(\phi_{t*}(\phi_{h*}) Y)_{\phi_{-t}(p)} - \phi_{t*} Y_{\phi_{-t}(p)}}{h} = 0$ by assumption of $[X, Y] = 0$. Thus $c(t) = c(0) = Y$. ☐

It is easy to see that $[e_i, e_j] = 0$. The following argument shows that the converse is true.

Theorem 3.2. *If $X_1, \ldots, X_k$ are linearly independent in a neighbourhood of p and $[X_i, X_j] = 0 \ \forall\ i, j$, then there is a coordinate system around p such that $X_i = e_i$.*

Proof. Call the flow of X_α as ϕ_α. After a linear change of coordinates we may assume that $X_i(p) = \frac{\partial}{\partial t^i}(p)$. Now consider the map $\chi(x^1, \ldots, x^k, y^{k+1}, \ldots, y^m) = \phi_{1,x^1}\phi_{2,x^2}\ldots\phi_{k,x^k}(0,0,0\ldots, y^{k+1}, \ldots y^m)$. As before we can use the IFT to show that this is a local diffeomorphism. Moreover, taking derivatives (and here is where we use the fact that the flows commute because the vector fields do), we see that $X_i = \frac{\partial}{\partial x^i}$. □

Next we prove that $L_X Y$ is a quantitative obstruction for the flows to commute.

Theorem 3.3. *If $c(t) = \psi_{-t} \circ \phi_{-t} \circ \psi_t \circ \phi_t(p)$. Then $c^{'}(0) = 0$ and $c^{''}(0) = 2[X, Y]_p$.*

Proof. Choose local coordinates such that $X = \frac{\partial}{\partial x^1}$ in a neighbourhood of p. Now

$$c^{'}(t) = \frac{\partial \psi_{-t}}{\partial t}(\ldots) + \frac{\partial \psi_{-t}}{\partial x^j}\frac{\partial \phi^j_{-t}}{\partial t} + \frac{\partial \psi_{-t}}{\partial x^j}\frac{\partial \phi^j_{-t}}{\partial x^k}\frac{\partial \psi^k_t}{\partial t} + \frac{\partial \psi_{-t}}{\partial x^j}\frac{\partial \phi^j_{-t}}{\partial x^k}\frac{\partial \psi^k_t}{\partial x^l}\frac{\partial \phi^l_t}{\partial t}$$

$$= -Y(\phi_{-t}\ldots) - \frac{\partial \psi_{-t}}{\partial x^j}X^j(\psi_t\ldots) + \frac{\partial \psi_{-t}}{\partial x^j}\frac{\partial \phi^j_{-t}}{\partial x^k}Y^k(\phi_t) + \frac{\partial \psi_{-t}}{\partial x^j}\frac{\partial \phi^j_{-t}}{\partial x^k}\frac{\partial \psi^k_t}{\partial x^l}X^l$$

$$c^{'}(0) = -Y_p - X_p + Y_p + X_p = 0$$

$$c^{''}(0) = \frac{\partial Y}{\partial x^j}X^j - \frac{\partial Y}{\partial x^j}Y^j - \frac{\partial Y}{\partial x^j}X^j + \frac{\partial Y}{\partial x^j}X^j - \frac{\partial X}{\partial x^k}Y^k - \frac{\partial X}{\partial x^k}X^k + \frac{\partial Y}{\partial x^j}X^j + \ldots$$

$$= 2[X, Y] \tag{3.3}$$

□