NOTES FOR 13 NOV (MONDAY)

1. Recap

(1) Proved a part of a theorem.

2. De Rham cohomology

Theorem 2.1. *If M is a connected orientable m-manifold, then the map $T[\omega] = \int_M \omega$ gives an isomorphism $H_c^m(M) \simeq \mathbb{R}$.*

Proof. (1) True on $\mathbb{R}$

(2) True on $\mathbb{R}^m$ assuming that it is true on S^{m-1}

(3) True for a general M assuming it is true for $\mathbb{R}^m$: Choose an m-form α such that $\int_M \alpha \neq 0$ and α is compactly supported on a coordinate open set U. If ω is any other m-form with compact support, then we want to show that $\omega = c\alpha + d\eta$ where η is compactly supported. Using a partition-of-unity, $\omega = \rho_1\omega + \rho_2\omega + \ldots$ where $\rho_i\omega$ is compactly supported in a coordinate chart U_i. It suffices to show that $\rho_i\omega = c_i\alpha + d\eta_i$ for each i (where η_i has compact support). So we can assume without loss of generality that ω is compactly supported on a coordinate chart V.

Let $V_1 = U$ where α has compact support. Unfortunately, we cannot use step 2) to conclude that $\omega = c\alpha + d\eta$ because α could be 0 on V. However, using connectedness, it is easy to see that we may assume that there are coordinate charts $V_1 = U, V_2, \ldots, V_r = V$ (all diffeomorphic to $\mathbb{R}^m$) such that $V_i \cap V_{i+1} \neq \phi$. Choose forms ω_i with supports in $V_i \cap V_{i+1}$ and $\int_{V_i} \omega_i \neq 0$. Using step 2, $\omega_1 = c_1\alpha + d\eta_1$, $\omega_2 = c_2\omega_1 + d\eta_2$, and so on. Therefore, $\omega = c\alpha + d\eta$. □

The method above can be used to derive other results.

Theorem 2.2. *If M is any connected non-orientable manifold, then $H_c^m(M) = 0$.*

Proof. Choose an m-form ω with compact support in a coordinate chart U (diffeomorphic to $\mathbb{R}^m$) such that $\int_U \omega \neq 0$ (the integral makes sense on U). It suffices to show that $\omega = d\eta$ for some compactly supported η.

Consider a sequence of coordinate systems $V_1 = U, V_2, \ldots, V_r$ with $V_i \cap V_{i+1} \neq 0$. Choose the forms ω_i in the earlier proof so that, using the standard coordinate orientations on V_i, $\int_{V_i} \omega_i > 0$ (therefore, $\int_{V_{i+1}} \omega_i > 0$). Consequently, $c_i = \int_{V_i} \omega_i / \int_{V_i} \omega_{i-1}$ are positive. Therefore, $\omega_i = c\omega + d\eta$ where $c > 0$. Since M is not orientable, there exists a sequence of coordinate charts such that $V_r \cap V_1 \neq 0$ and $x_r \circ x_1^{-1}$ is orientation-reversing. (The HW asks you to prove that every manifold is a union sequence of successively intersecting coordinate charts such that eventually, the sequence lies outside every compact set. Therefore, if for every closed curve, the corresponding sequence of sets intersecting it are orientation-preserving, then the manifold has been equipped with an orientation by charts.) Hence $-\omega = c\omega + d\eta$ where $c > 0$. Thus $(-c-1)\omega = d\eta$ where $-c-1 \neq 0$. □

The same technique actually allows us to calculate $H^m(M)$ for non-compact manifolds too.

Theorem 2.3. *If M is a connected non-compact manifold (orientable or not), then $H^n(M) = 0$.*

Proof. Since M is not compact, the HW problem tells us that there is an infinite sequence of coordinate neighbourhoods (whose closures are compact) $U_1, U_2, \ldots$ such that $U_i \cap U_{i+1} \neq \phi$ such that the sequence lies eventually in the complement of any compact set. It also says that the union of these sets is all of M. The cover is locally finite. Choose a partition-of-unity subordinate to the cover. Now $\omega = \sum \phi_{U_i}\omega$. Thus it is enough to prove the result when ω is supported in U_1.

Choose m-forms ω_i with compact support in $U_i \cap U_{i+1}$ such that $\int_{U_i} \omega_i \neq 0$. Then $\omega = c\omega_1 + d\eta_1$ (where η_i have compact support) and $\omega_i = c_{i+1}\omega_{i+1} + d\eta_{i+1}$. Thus, $\omega = d\eta_1 + c_1 d\eta_2 + \ldots$. Since every point p is eventually in the complement of the U_i, we see that the right hand hand side makes sense (it is not an infinite sum). ☐

To summarise, we computed $H^k(\mathbb{R}^m)$, $H^m_c(M)$, $H^m(M)$. Recall that if f is a diffeomorphism between manifolds, and ω is a compactly supported top form, then $\displaystyle\int f^*\omega = \pm \int \omega$ depending on whether f is orientation-preserving or not. Actually, this can be generalised.

Suppose $f : M \to N$ is a proper map ($f^{-1}(\textit{compact}) = \textit{compact}$). Suppose ω_0 is a top form on N with non-zero integral. Also assume that $\int_M f^*\omega_0 = a \int_N \omega_0$. Then, because the integration map is an isomorphism to $\mathbb{R}$, $\int_M f^*\omega = a \int_N \omega$ for any compactly supported form ω. Shockingly enough, a is always an integer (called "the degree of f").