NOTES FOR 13 OCT (FRIDAY)

1. Recap

(1) Every subalgebra of the Lie algebra of a Lie group is the Lie algebra of a Lie subgroup.
(2) A Lie algebra homomorphism between Lie algebras of Lie groups only *locally* induces a Lie group homomorphism.
(3) If two Lie algebras (of Lie groups) are isomorphic, then their corresponding Lie groups are locally isomorphic.
(4) A connected Lie group with an abelian Lie algebra is abelian.
(5) There is a unique one-parameter subgroup through the identity with tangent vector $X_e \in \mathfrak{g}$. (The exponential map is $\exp(X_e) = \phi(1)$ where ϕ is that one-parameter subgroup.)

2. Lie algebras (cont'd..)

Proposition 2.1. *The exponential map is smooth and a diffeomorphism around* 0. *If* $\psi : G \to H$ *is any smooth homomorphism, then* $\exp \circ \psi_* = \psi \circ \exp$.

Proof. The easiest way to prove smoothness is by using the fact that the flow of a vector field on a manifold is smooth in both, the time, and the space parameters. Indeed, define a vector field Y on $T_eG = \mathbb{R}^m \times G$ by $Y(X_e, a) = 0 \oplus L_{a*}X_e$. Now Y is a smooth vector field (why? one way is to write it down in coordinates. We already saw that $L_{a*}B$ is a smooth in a if B is fixed. Y is linear in X_e. I leave the details to you). Therefore it has a smooth flow $\alpha : T_eG \times G \times \mathbb{R} \to T_eG \times G$. Now $\exp(X) = \pi_2 \circ \alpha(X_e, 0, 1)$ which is smooth.

We shall prove that $\exp_*$ at the identity is simply the identity map (which is an isomorphism). By the IFT exp is a local diffeomorphism. Indeed, suppose $c(t) = tv$ is a curve on T_eG, then in note that $\exp(c(t)) = \exp(tv)$ is the time-1 flow of the vector field $X_t = tX$. I claim that this is the same as the time-t flow of X. Indeed, define $\psi(s) = \phi(ts)$. Then $\frac{d\psi}{ds} = t\phi'(ts) = tX$. Therefore, $\frac{d\exp(tv)}{dt}|_{t=0} = v$.

Suppose $X_e \in T_eG$. Then $\tilde{X}(b) = L_{b*}(\psi_* X_e)$ is the left-invariant extension on H. I claim that $\psi(\phi(t))$ where $\phi(t)$ is the flow of $X(a) = L_{a*}X_e$ on G is the flow through identity of $\tilde{X}$ on H. Indeed, $\psi_*\phi_*(\frac{\partial}{\partial t}) = \psi_* X(\phi(t)) = \psi_* L_{\phi(t)*}X_e = L_{\psi(\phi(t))*}\psi_* X_e = \tilde{X}$. The last equality holds because $\psi(L_a b) = \psi(ab) = \psi(a)\psi(b) = L_{\psi(a)}\psi(b)$. $\square$

As a corollary,

Corollary 2.2. *Every* $1-1$ *smooth homomorphism of Lie groups* $h : G \to H$ *is an immersion.*

Proof. Suppose at a point $p \in G$, there is a vector $v \in T_pG$ such that $h_*v = 0$. Then $h_* L_{p*}(v^i X_{e,i}) = 0$ where $X_{e,i}$ is some basis of T_eG. Because h is a homomorphism, $L_{h(p)*}h_*(v^i X_{e,i}) = 0$. This means that h_* has a kernel at the identity. So $e = \exp(h_*(tv_e)) = h(\exp(tv_e))$ which contradicts the injectivity of h. $\square$

A more interesting corollary is that

Corollary 2.3. *Every continuous homomorphism* $\phi : \mathbb{R} \to G$ *is smooth.*

This can be generalised to

Corollary 2.4. *Every continuous homomorphism $\phi : G \to H$ is smooth.*

We shall omit the proofs of these corollaries. (The rough idea for the first is to show that $\phi(s\epsilon) = \exp(sX)$ for some ϵ and X and use the smoothness of exp. For the second, you use a basis of $T_e G$ and use the first.) Therefore, if two Lie groups are isomorphic by a homeomorphic isomorphism then there is also a diffeomorphic isomorphism.

3. Tensors and tensor fields

Recall that a vector field is a section of the tangent bundle TM. Also, TM as a set is $\cup_p T_p M$ (where $T_p M$ can be defined either as the equivalence class of curves or as point-derivations on the germs of smooth functions). It can be made into a smooth vector bundle of rank-m over M by simply using the point derivations $\frac{\partial}{\partial x^i}$ for a coordinate chart (x, U) to produce local trivialisations. Every smooth vector field X is locally of the form $X = X^i \frac{\partial}{\partial x^i}$ where X^i are local smooth functions. When you change coordinates to $(\tilde{x}, U)$, the new components $\tilde{X}^i = X^j \frac{\partial \tilde{x}^i}{\partial x^j}$.

Recall that while we define the cotangent bundle T^*M in a complicated, weird manner earlier, more succinctly, it is simply the dual bundle of the tangent bundle, i.e., as a set it is $\cup_p T_p^* M$ where $T_p^* M$ is the dual of $T_p M$ (i.e. it consists of linear maps from $T_p M$ to $\mathbb{R}$). The elements of $T_p^* M$ are called one-forms at p. Given a coordinate neighbourhood, there are "dual" one-forms dx^i to $\frac{\partial}{\partial x^i}$. They satisfy $dx^i(X) = X^i$. It can be made into a vector bundle using the trivialisations given by the dx^i. Alternatively, if $g_{\alpha\beta}$ are the transition functions of a vector bundle V, then $[g_{\alpha\beta}^{-1}]^T$ are those of the dual vector bundle V^*. Just as sections of TM are called vector fields, the sections of T^*M should be called "one-form fields" but we abused terminology and called them one-forms. Every one-form field $\omega = \omega_i dx^i$ locally where ω_i are locally smooth functions that transform as $\tilde{\omega}^i = \omega^j \frac{\partial \tilde{x}^i}{\partial x^j}$. Given a one-form and a vector field, one can come up with a function $p \to \omega_p(X_p)$.

Given a smooth map $f : M \to N$, we came up with $f_* : T_p M \to T_{f(p)} N$. However, we discussed that it does not make sense to pushforward vector fields in general. On the other hand, there is a way to pullback one-forms (and even one-form fields) $f^* : TN^* \to TM^*$, i.e., $f^*(\omega_p)(X_p) = \omega_p(f_* X_p)$. At the level of coordinates, $f_* v$ is simply $[Df]\vec{v}$ where $\vec{v}$ is a column vector and $[Df]$ is the derivative matrix (whose rows are the gradients of the component functions). On the other hand, $f^*\omega = \omega[Df]$ where ω is a row vector.

One can construct lots of examples of vector fields and one-forms using bump functions and partitions-of-unity. One reason to care about vector fields is to construct diffeomorphisms. We have not shown a reason to care about one-forms yet (except perhaps to differentiate functions). (Spoiler alert : The point is to formulate a fundamental theorem of calculus called Stokes' theorem which can in turn be useful while trying to solve PDE on manifolds just like Green's theorem can be used to study the Laplace equation on $\mathbb{R}^n$. Another related point is that the cohomology of manifolds has a particularly nice model using forms.) Lastly, given a smooth function $f : M \to \mathbb{R}$, there is anice way to construct a 1-form field : $df_p(X_p) = X_p(f)$ or more simply, in coordinates, $df = \frac{\partial f}{\partial x^i} dx^i$.

The notation dx^i, df should remind you of "classical" nonsense like "infinitesimals" that Leibniz, Newton, and some misguided physicists use. This is the closest we can get to making a statement like $\Delta f = f' \Delta x$ precise. Indeed, df is supposed to detect small changes in f, so it is supposed to measure the directional derivative along a specified direction v. Thus it is morally supposed to be a 1-form. So, indeed, just as expected, $\Delta f = \frac{\partial f}{\partial x^i} \Delta x^i$ and $df = \frac{\partial f}{\partial x^i} dx^i$. Classically, tangent vectors used to be called "contravariant vectors" and cotangent vectors as "covariant vectors".