NOTES FOR 13 SEPT (WEDNESDAY)

1. Recap

(1) Proved that $L_X Y = [X, Y]$. Defined Lie algebras.
(2) Proved flows commute if and only if vector fields do.
(3) $X_1, \ldots X_k$ are locally coordinate vector fields if and only if they commute.
(4) Presented the Lie derivative as a quantitative second order obstruction for flows to commute.

2. Lie bracket, Frobenius theorem

Now that we saw the Lie bracket is the obstruction for a bunch of linearly independent vector fields $X_1, \ldots, X_k$ to be coordinate vector fields, we can ask a weaker question : Suppose we are given a bunch of vector fields $X_1, \ldots, X_k$ that are linearly independent in a neighbourhood U of a point p, is there a k-dimensional submanifold $S \subset U$ such that $X_1, \ldots, X_k$ form a basis for $T_q S$ for all $q \in U$?

This is not always true. Indeed, suppose we look at two vector fields in $\mathbb{R}^3$ given by $X_1 = (1, 0, y)$ and $X_2 = (0, 1, 0)$. Suppose there is a 2-dimensional submanifold S near the origin with coordinates (u, v) such that $X_1 = a(u, v)\frac{\partial}{\partial u} + b(u, v)\frac{\partial}{\partial v}$ and $X_2 = c(u, v)\frac{\partial}{\partial u} + d(u, v)\frac{\partial}{\partial v}$. Actually, since X_1 and X_2 do not point along $\hat{k}$ at the origin, we can assume wlog that $u = x$, $v = y$ and S to be a graph $(x, y, z(x, y))$. Now $i_*(\frac{\partial}{\partial x}) = \frac{\partial}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z}$ and $i_*(\frac{\partial}{\partial y}) = \frac{\partial}{\partial y} + \frac{\partial z}{\partial y}\frac{\partial}{\partial z}$. Thus we want to find four functions a, b, c, d satisfying $a = 1, a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = y, b = 0, c = 0, d = 1, c\frac{\partial z}{\partial x} + d\frac{\partial z}{\partial y} = 0$. Thus $\frac{\partial z}{\partial x} = y$ and $\frac{\partial z}{\partial y} = 0$. Thus $\frac{\partial^2 z}{\partial x \partial y} = 0 \neq 1 = \frac{\partial^2 z}{\partial y \partial x}$.

The moral of the above story is that a *necessary* condition among X_1 and X_2 which should eventually boil down to mixed partial derivatives of somethings being equal is not met. The Frobenius integrability theorem states very roughly that if such natural necessary conditions are met, we can find submanifolds "tangent" to the given vector fields.

More generally, we define a "k-dimensional distribution on M" (not to be confused with distributions occurring in analysis) or perhaps in more modern terminology, a "rank-k subbundle of TM" has a collection of subspaces $\Delta_p \subset T_p M$ for all $p \in M$ such that for every $p \in M$ there is a neighbourhood U and k smooth vector fields $X_1, \ldots, X_k$ such that $X_1(q), \ldots, X_k(q)$ form a basis for Δ_q. A k-dimensional submanifold N is called an integral manifold of Δ if for every $p \in N$ we have $i_*(T_p N) = \Delta_p$ where i is the inclusion map. To kill the suspense, more often than not, integral manifolds are not embedded submanifolds. But immersed submanifolds exist (whenever certain necessary conditions are met). But we will not go into the details of the global phenomena (which deal with things called foliations). Instead we will state the Frobenius integrability theorem locally.

Note that if an integral submanifold exists and $X_1, \ldots, X_k$ locally span the given subbundle, then $[X_i, X_j]$ ought to lie in the tangent space of the submanifold and hence in the distribution. Therefore, $[X_i, X_j] = C_{ij}^k X_k$ for some smooth functions C_{ij}^k. This is a necessary condition and if this is met, the distribution is said to be "integrable". (Note that the above example is not integrable.) The Frobenius theorem states that locally, this is sufficient as well.

Theorem 2.1. *Let Δ be an integrable k-dimensional distribution on M. For every $p \in M$, there is a coordinate system (x, U) with $x(p) = 0$, $x(U) = (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \ldots$, such that for each $a^{k+1}, \ldots, a^m$ with $|a^i| < \epsilon$, the set $\{q \in U : x^{k+1}(q) = a^{k+1} \ldots x^m(q) = a^m\}$ is an integral manifold of Δ.*

The proof is based on using the flows of the vector fields. We will omit it. You can read it in Spivak.