NOTES FOR 15 NOV (WEDNESDAY)

1. Recap

(1) Computed the De Rham groups $H^k(\mathbb{R}^m), H^m_c(M), H^m(M)$.
(2) Defined the degree of a proper map.

2. Degree of proper maps

Theorem 2.1. *If $f : (M, \mu) \to (N, \nu)$ is a proper map between oriented manifolds of the same dimension, then $f^* : H^m(N) \to H^m(M)$ given by $\int_M f^*\omega = deg(f) \int_N \omega$ satisfies the following : If $q \in N$ is any regular value, then $deg(f) = \sum_{p \in f^{-1}(q)} (-1)^{k_p}$ where k_p is even if $f_{*p} : T_pM \to T_qN$ is orientation-preserving and odd otherwise.*

Proof. Note that if $p \in f^{-1}(q)$, by the inverse function theorem, f is a local diffeomorphism $f : p \in U_p \subset M \to V_p \subset N$ where U_p is a coordinate neighbourhood of p. Thus $f^{-1}(q)$ is covered by disjoint open sets. By properness, $f^{-1}(q) = \{p_1, \ldots, p_j\}$ is a finite set and $V = V_1 \cap V_2 \ldots$ is a neighbourhood of q such that $f^{-1}(V)$ is a finite union of open neighbourhoods (we which continue calling U_i abusing notation a bit) of p_i. Choose a top form ω on N compactly supported on V, compatible with the orientation, and $\int_N \omega \neq 0$. Now, $\int f^*\omega = \sum_{U_i} \int f^*\omega = \sum_i \pm 1$ depending on whether f is orientation preserving at p_i or not. □

Corollary 2.2. *A smooth proper map between orientable manifolds of the same dimension is a finite-sheeted covering map when restricted to regular points.*

Let us calculate the degree of the antipodal map $A : S^n \to S^n$ given by $A(p) = -p$. Clearly this is a diffeomorphism. $A_* = -Id$. It is orientation-preserving when n is odd (indeed, the normal N at p goes to $-N$ at $-p$ which is the correct normal. So it is orientation preserving on S^n if and only if it is so on $\mathbb{R}^{n+1}$. Its determinant on $\mathbb{R}^{n+1}$ is $(-1)^{n+1}$.) and reversing otherwise. Thus the degree is 1 for odd n and -1 otherwise. Before applying this observation (to prove the Hairy Ball theorem), we prove the following important result.

Theorem 2.3. *If $f, g : M \to N$ are smoothly homotopic, i.e., there exists a smooth $H : M \times [0, 1] \to N$ such that $H(1, p) = f(p)$ and $H(0, p) = g(p)$, then $f^* = g^*$ on De Rham cohomology.*

Proof. Suppose $\omega \in \Omega^k(N)$ is a representative of $[\omega]$. Then, $i_1^* H^*\omega - i_0^* H^*\omega = dP\omega + Pd\omega = dP\omega$. Therefore, $f^*[\omega] = g^*[\omega]$. □

Corollary 2.4. *If M and N are compact oriented manifolds of the same dimension, and $f, g : M \to N$ are smoothly homotopic, then $deg(f) = deg(g)$.*

Corollary 2.5. *If M smoothly deformation retracts to N, i.e., $H : M \times [0, 1] \to M$ satisfies $H(p, 1) = r(p)$, $H(p, 0) = p$ where $r : M \to N \subset M$ is a retraction, then $H^k(M) = H^k(N)$.*

Proof. Let $i : N \to M$ be the inclusion map. Note that $r \circ i = Id$ (therefore $i^* \circ r^* = Id$). Also, $i \circ r : M \to M$ is homotopic to the identity and hence $r^* \circ i^* = Id$. □

Corollary 2.6. *If n is even, then there is no nowhere vanishing vector field on S^n.*

Proof. Note that the antipodal map $A : S^n \to S^n$ is not homotopic to the identity when n is even (their degrees do not match). Suppose there is such an $X : S^n \to TS^n$. Then, at p, let $\gamma_p(t)$ be the unique great semicircle (depending smoothly on p) taking p to $A(p) = -p$ and whose tangent vector at p is a multiple of $X(p)$. Define $H(p, t) = \gamma_p(t)$. □

For odd n, there is an explicit nowhere vanishing vector field $X = (-x^1, x^0, -x^3, x^2, \ldots)$.

Corollary 2.7. $H^k(\mathbb{R}^{n+1} - 0) = H^k(S^n)$.

Corollary 2.8. $H^k(M \times \mathbb{R}^k) = H^k(M)$.

Some thing slightly harder is $H^k(V) = H^k(M)$ where V is a vector bundle over M.