NOTES FOR 15 SEPT (FRIDAY)

1. Recap

(1) Defined distributions, integrable distributions, subbundles, and integral submanifolds.
(2) Stated Frobenius' theorem that integrable distributions locally look standard and admit integral submanifolds locally. (Mumbled about the integral submanifolds being only immersed submanifolds globally.)

2. Lie groups and Lie algebras

Sophius Lie was studying systems of differential equations having lots of symmetries. Inspired by Galois theory (where discrete symmetries of a polynomial equation could be exploited (in some cases) to solve the equation), Lie set out to see whether continuous symmetries of ordinary differential equations can sometimes be used to solve the equations themselves. More generally, continuous symmetries of PDE can be used to solve the PDE. (In quantum mechanics, the hydrogen atom and the harmonic oscillator are prime examples.) More information is in http://www.physics.drexel.edu/~bob/LieGroups/LG_16.pdf.

So the study of "continuous groups", i.e. is there a "list" of "standard continuous groups" ? is obviously important. To make this notion precise, we define Lie groups : A Lie group G is a manifold having a smooth map $\times : G \times G \to G$ obeying

(1) There is an element (identity) id such that $a \times id = id \times a = a$.
(2) For every a, there is an inverse a^{-1} such that $a^{-1} \times a = a \times a^{-1} = id$. Moreover, $a \to a^{-1}$ is smooth (and hence a diffeomorphism).
(3) $a \times (b \times c) = (a \times b) \times c$.

It is immediately clear that $G \times H$ is a Lie group whenever G and H are. A Lie group homomorphism is a smooth map which is also a group homomorphism. It is also clear that the projection maps from $G \times H$ to G and H are Lie group homomorphisms. Here are examples of Lie groups :

(1) $(\mathbb{R}^n, +)$.
(2) S^1. Indeed, consider $S^1 \subset \mathbb{R}^2$ and define the smooth map from $\mathbb{R}^4$ to $\mathbb{R}^2$ given by $z_1 z_2$. This restricts to a smooth map $S^1 \times S^1 \to S^1$. It is easy to show that $z \to \frac{1}{z}$ is smooth. (Alternatively, you can think of S^1 as $\mathbb{R}Z$ by $f(x) = e^{2\pi\sqrt{-1}x}$.) Therefore the n-torus $S^1 \times S^1 \ldots$ is a Lie group $\mathbb{R}^n/\mathbb{Z}^n$.
(3) $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$ are Lie groups under matrix multiplication (which is smooth because polynomials are so; matrix inversion is smooth because rational functions are so on their domains).
(4) We proved that $O(n, \mathbb{R})$ and $U(n, \mathbb{C})$ are smooth manifolds. They are also Lie groups under matrix multiplication.
(5) It is not hard to prove the symplectic group $Sp(2n, \mathbb{R})$ consisting of $2n \times 2n$ invertible matrices A such that $A^T J A = J$ where

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \tag{2.1}$$

is a Lie group. Indeed, it is enough to prove that it is an embedded submanifold of $GL(2n, \mathbb{R})$. (It is clearly closed under multiplication.) Consider $f(A) = A^T JA$ from $GL(n, \mathbb{R})$ to the submanifold of $GL(n, \mathbb{R})$ consisting of skew-symmetric matrices (why is this a submanifold ? what is its tangent space at any point ?). We need to prove that J is a regular value, i.e., if $A_0 \in Sp(2n, \mathbb{R})$, then for every skew-symmetric matrix B, there exists a curve of invertible matrices $A(t)$ such that $\frac{df(A(t))}{dt}|_{t=0} = B$ and $A(0) = A_0$. Indeed, if $A(t)$ is any curve through A_0, then

$$\frac{df(A)}{dt}|_{t=0} = \frac{d}{dt}|_{t=0}(A^T JA - J) = (A^{'}(0))^T JA_0 + A_0 JA^{'}(0) \tag{2.2}$$

Define $A^{'}(0) = -\frac{1}{2}JA_0^{-1}B$ (and hence choose $A(t) = A_0 + tA^{'}(0)$). This does the job. (Why?)