NOTES FOR 16 AUG (WEDNESDAY)

1. Maps to manifolds and all that

Given a function $F : \mathbb{R}^m \to \mathbb{R}$ such that for all point $\vec{x}$ on the level set L given by $F(\vec{x}) = 0$, the function satisfies $\nabla F(\vec{x}) \neq \vec{0}$, then $L \subset \mathbb{R}^m$ is an embedded submanifold !

Proof :Indeed, suppose we take a point $p \in L$. Assume that $\frac{\partial F}{\partial x^1}(p) \neq 0$ (without loss of generality). Then by the IFT, $x^1 = g(x^2, \ldots, x^m)$ locally on an open subset $V \subset \mathbb{R}^m$ for some smooth function $g : U \subset \mathbb{R}^{m-1} \to \mathbb{R}$. Thus, $\Phi_V : V \cap L \to \mathbb{R}^{m-1}$ given by $(x^1, \ldots, x^m) \to (x^2, \ldots, x^m)$ is a coordinate chart. If you take another such coordinate chart, where this time $x^2 = \tilde{g}(x^1, x^3, \ldots)$, then the transition function is $(x^2, \ldots, x^m) \to (g(x_2, \ldots, x_m), x_3, \ldots)$ which is smooth and whose inverse is $(x^1, x^3, \ldots) \to (\tilde{g}, x^3, \ldots)$. So L is a smooth manifold. Moreover, $i : L \to \mathbb{R}^m$ is an immersion because in local coordinates it looks like $(x^2, \ldots, x^m) \to (g(x^2, \ldots), x^2, \ldots)$ whose derivative is

$$\begin{bmatrix} \frac{\partial g}{\partial x^2} & \frac{\partial g}{\partial x^3} & \ldots \\ 1 & 0 & \ldots \\ 0 & 1 & 0\ldots \\ \vdots & \vdots & \ddots \end{bmatrix} \tag{1.1}$$

which is of course injective as a linear map. Now L is homeomorphic to $i(L)$ because its topology is derived from $i(L)$. □

More generally, one can prove that if a level set $L = \vec{F}^{-1}(\vec{a})$ of a smooth function $\vec{F} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ consists of only regular points (i.e. $\vec{a}$ is a regular value), then indeed L is an embedded submanifold of $\mathbb{R}^{m+n}$ of dimension m. This is the point of defining regular values (their pre-images are embedded submanifolds).

Harish (I believe) proved that $O(n)$ is a manifold using the above observation. Here are a couple of other examples.

(1) The sphere is a manifold : $f(x^1, \ldots, x^{n+1}) = \sum (x^i)^2 - 1$. Now $\nabla f = (2x^1, \ldots, 2x^{n+1})$ is never zero on the level set (because at least one of the $x^i \neq 0$). Thus the sphere is an n-dimensional submanifold of $\mathbb{R}^{n+1}$.
(2) The unitary group $U(n)$ is a submanifold of $\mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$. This will be a part of your HW.

Actually, the phenomenon above is even more general. To see this, we need the following result :

Theorem 1.1 (Constant rank theorem)**.** *If $f : M^m \to N^n$ is a smooth map between smooth manifolds, such that the rank of f in a neighbourhood of p is a constant equal to k, then there exist coordinate charts $(\Phi_U = x, p \in U)$, $(\Phi_{\tilde{U}} = \tilde{x}, f(p) \in \tilde{U})$ around p, $f(p)$ respectively such that*

$$\Phi_{\tilde{U}} \circ f \circ \Phi_U^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^k, 0, 0, \ldots) \tag{1.2}$$

(from now onwards, we will omit writing the Φ)

Proof. Indeed, by assumption in coordinate charts $(\Phi_V = y, p \in V)$, $(\tilde{\Phi}_{\tilde{V}} = \tilde{y}, \tilde{V})$ we see that the matrix of $D_y f$ in these coordinates has rank k in V. Therefore, after permuting the coordinates we can arrange that the first $k \times k$ piece of the derivative matrix $D_y f$, i.e., $\frac{\partial f^i}{\partial y^j}$ $1 \leq i, j \leq k$ is invertible at p. Thus, locally, in these coordinates, we have a map $F : \mathbb{R}^k \times \mathbb{R}^{m-k} \to \mathbb{R}^k$ given by $F(y) = (f^1, f^2, \ldots, f^k)$.

Consider the function $G : \mathbb{R}^m \to \mathbb{R}^m$ given by $G(y) = (F(y), y^{k+1}, y^{k+2}, \ldots, y^m)$. The derivative $DG(p)$ is invertible (why ?). By the inverse function theorem, G is a local diffeomorphism, i.e., $\vec{u} = (\tilde{y}^1, \tilde{y}^2, \ldots, \tilde{y}^k, y^{k+1}, \ldots, y^m)$ is a valid coordinate chart on M (because it is diffeomorphic to another valid one). In this new coordinate chart, the map f looks like $f(\vec{u}) = (u^1, u^2, \ldots, u^k, f^{k+1}, f^{k+2}, \ldots, f^n)$. This is almost what we want (we want to make the other $n - k$ coordinates zero). Notice that since the rank is exactly k in a neighbourhood of p, the functions $f^{k+1}, \ldots, f^n$ should not depend on $u^{k+1} = y^{k+1}, u^{k+2}, \ldots, u^m$. Now we define a map $H : \mathbb{R}^n \to \mathbb{R}^n$ given by $H(\vec{u}) = (u^1, \ldots, u^k, y^{k+1} - f^{k+1}(u^1, \ldots, u^k), y^{k+2} - f^{k+2}, \ldots)$. The derivative $DH_u(p)$ is invertible (why ?). By the inverse function theorem H is a local diffeomorphism, i.e., the functions $\vec{v} = (u^1, \ldots, u^k, y^{k+1} - f^{k+1}(u^1, \ldots, u^k), y^{k+2} - f^{k+2}, \ldots)$ form a coordinate chart for N around $f(p)$. In this new chart, the map f looks like $f(\vec{u}) = (u^1, \ldots, u^k, 0, 0 \ldots)$. This proves the desired result. ◻

Remark 1.2. If f has full rank at p (i.e. f is either an immersion or a submersion) then it has full rank in a neighbourhood of p (why ?). Thus, an immersion locally looks like $(x^1, \ldots, x^m) \to (x^1, \ldots, x^m, 0, 0 \ldots)$ and a submersion locally looks like $(x^1, \ldots, x^m) \to (x^1, \ldots, x^n)$.

Remark 1.3. In particular, from the previous remark, it follows that if S is an embedded submanifold of M, then locally there exist coordinates $(x^1, \ldots, x^m)$ in M such that $(x^1, \ldots, x^s)$ form coordinates for S.

From now onwards, whenever we say "submanifold", we mean "embedded submanifold". Here is a proposition that helps us construct lots of examples of submanifolds. (It is a generalisation of the previous method to construct submanifolds of $\mathbb{R}^n$ using level sets.) It's proof is left as an exercise (use the constant rank theorem).

Proposition 1.4. *if $f : M \to N$ is a C^∞ map having constant rank k on a neighbourhood of $f^{-1}(y)$, then $f^{-1}(y)$ is either*

(1) *Empty, or*
(2) *A closed submanifold of M of dimension $n - k$.*

In particular, if y is a regular value of f and $f^{-1}(y)$ is not empty, then it is an $n - m$-dimensional submanifold of M.

Here is another application of the constant rank theorem :

Proposition 1.5. *If $M_1 \subset M$ is an* immersed *submanifold, $f : P \to M$ is a smooth map from a smooth manifold P such that $f(P) \subset M_1$, and f is continuous is considered as a map into M_1, then f is also smooth as a map into M_1.*

Proof. Since M_1 is immersed in M, around a point $f(p) \in M_1$, there exist coordinates $(x^1, \ldots, x^m)$ on M and coordinates $(y^1, \ldots, y^{m_1})$ on M_1 such that the inclusion map is $i(y^1, \ldots, y^{m_1}) = (x^1 = y^1, \ldots, x^{m_1} = y^{m_1}, 0, \ldots)$. Let $U_1 \subset M$ be the set consisting of $x^i = 0 \; \forall \, i \geq m_1 + 1$. $i^{-1}(U_1)$ is an open subset of M_1. By assumption, $i^{-1} \circ f$ is continuous, so $f^{-1} \circ i(\mathit{open}) = \mathit{open}$. Therefore $f^{-1}(U_1) \subset P$ is open. Therefore, f takes a neighbourhood of $p \in P$ into U_1. Since $y^i \circ f$ are smooth, f is smooth considered as a map into M_1. ◻

2. Partition-of-unity, Whitney embedding

Now that we know how to come up with examples of manifolds as submanifolds of $\mathbb{R}^N$, it is but natural to ask 2 questions :

(1) (*Embeddability*) Is every smooth manifold secretly a submanifold of $\mathbb{R}^N$? i.e., for every M is there a smooth embedding $f : M \to \mathbb{R}^N$?
(2) (*Global intersection*) Is every k-dimensional submanifold of $\mathbb{R}^N$ the zero level set of a smooth function $\vec{F} : \mathbb{R}^N \to \mathbb{R}^{N-k}$ such that 0 is a regular value of $\vec{F}$?

To kill the suspense, the answer to the first question is YES. In fact, we can choose N to be $2n$ (but this is the best one can do because one can prove that $\mathbb{RP}^{2^q}$ cannot be embedded in $\mathbb{R}^{2^{q+1}-1}$). The answer to the second is NO. Under some assumptions, the answer to second is also YES but in general (even after imposing some necessary conditions), it seems to be open (at least as far as I know). To kill whatever little suspense is remaining, the answer to the first question is provided by the Whitney embedding theorem. The second question has some obstructions. They can be studied using vector bundles (in particular, the normal bundle). In fact, one of the points of studying vector bundles is to answer questions like the second one. But we are getting far ahead of ourselves.

Bump functions are nice tools but we need something better in order to prove things like the Whitney embedding theorem. The topology necessary to develop good enough bump functions to prove Whitney allows us to develop another (unrelated) nice tool - partition-of-unity. There are two definitions of a partition-of-unity. Both are useful in different circumstances.

Definition 2.1. Given a locally finite open cover $\{U_i\}$ (where $i \in I$) of a manifold M, a collection of smooth functions $\rho_i : M \to [0, 1]$ such that $supp(\phi_i) \subset U_i$ satisfying $\sum_i \phi_i = 1$ is called a partition-of-unity subordinate to the open cover $\{U_i\}$.

Locally finite means that every point $p \in M$ is in only finitely many U_i. On a paracompact space, every open cover U_α has a locally finite refinement, i.e., there is another locally finite cover V_β such that every open set V_β is in *some* U_α (but may be there are fewer V_β than U_α or perhaps even more). Recall that our manifolds are required to be paracompact by definition. This gives rise to another definition.

Definition 2.2. Given an arbitrary open cover U_i ($i \in I$) of a manifold M, a collection of smooth *compactly supported* functions $\rho_j : M \to [0, 1]$ (over a possibly distinct index set $j \in J$) such that $supp(\phi_j) \subset U_i$ for some i and for every point $p \in M$ at most finitely many $\phi_j(p) \neq 0$, satisfying $\sum_i \phi_i = 1$ is called a partition-of-unity subordinate to the open cover $\{U_i\}$.

Usually, when someone says "partition-of-unity subordinate to an open cover", one means the second definition. (Some people do not even bother imposing compact support in the second definition, but we might as well do so.)