NOTES FOR 16 OCT (MONDAY)

1. Recap

(1) Proved that the exponential map is smooth, is a local diffeomorphism near 0 and it "commutes" (in a sense) with Lie group homomorphisms.
(2) Stated some interesting corollaries - Every injective homomorphism is an immersion, and every continuous homomorphism is smooth.
(3) Revised tangent vectors, vector fields, one-forms, one-form fields (which we shall continue calling one-forms), etc.

2. Tensors and tensor fields

In physics and engineering, very often things are not simply vectors. For example, the moment of inertia of a very asymmetric object or the "stress" experienced by such an object under a force are morally to be thought of as matrices, i.e., objects that have two indices like I_{ij}, T_{ij} etc. What if tomorrow we find objects that need three indices ? How can we make sense of them on manifolds ?

Even in pure mathematics, given a vector space V, one natural construction (i.e. a functor from $Vect$ to itself) is V^*. But there are other constructions of new vector spaces like $V \times V$ for instance. If you have a vector bundle V over M, you can create V^* by taking the new transition functions as $([g_{\alpha\beta}^{-1}]^T$. If you have W also, then you can create $V \oplus W$ whose transition functions are $g_{\alpha\beta} \oplus h_{\alpha\beta}$ where $\oplus$ is the direct sum of matrices. Now if you look at multilinear maps $V \times V \times V \ldots \to \mathbb{R}$ (there are natural examples of these - like the determinant and the permanent for instance) then it is not obvious to see what the dimension of the space of multilinear maps is. To remedy this, recall that one constructs a nice vector space called $V \otimes V \otimes \ldots$ such that every multilinear map actually factors uniquely as a linear map from this "tensor product". Elements of tensor product spaces are called tensors. So, corresponding to two vector bundles V and W, we can construct a tensor product bundle as $V \otimes W$ having transition functions $[g_{\alpha\beta}] \otimes [h_{\alpha\beta}]$ where $\otimes$ on matrices refers to the Kronecker product.

So we can inductively look at the vector bundles $TM \otimes TM \otimes TM \ldots$ or $T^*M \otimes T^*M \otimes T^*M \ldots$ or more generally $TM \otimes TM \otimes \ldots TM \otimes T^*M \otimes T^*M \ldots$. The elements of these vector spaces are called tensors and sections of these bundles are called tensor fields or abusing terminology, simply tensors. (Sections of the tensor product of only TM are called contravariant tensors. Those of purely T^*M are called covariant tensors and those of a mixture of them are called mixed tensors.)

Let's look at them locally and in more detail. Take $TM \otimes TM$. While we can construct this bundle using transition functions, let us take a slightly different approach (which is reminiscent of our construction of TM). As a set it is $\cup_p T_pM \otimes T_pM$. We make it into a vector bundle by providing local trivialisations : Take any coordinate chart (x, U). Then the vectors $\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$ span $T_qM \otimes T_qM$ for all $q \in U$. Thus we can get a bijection from $U \times \mathbb{R}^m \otimes \mathbb{R}^m$ to $\cup_{p \in U} T_pM \otimes T_pM$. These subsets are declared to form a basis for the topology. Since U are coordinate neighbourhoods, we get a locally Euclidean structure such that transition maps are smooth. In fact, this is a vector bundle. The point is that every element in the fibre looks like $A^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$ locally. These are tensors. Smooth sections of $TM \otimes TM$ are called "rank-2 contravariant tensor fields". They are locally of the form $A^{ij}(x)\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j}$ where the $A^{ij}(x)$ are smooth. If you change coordinates to $(\tilde{x}, U)$, then $\tilde{A}^{ij} = A^{kl}\frac{\partial \tilde{x}^i}{\partial x^k}\frac{\partial \tilde{x}^j}{\partial x^l}$. More generally, the same story makes sense for $TM \otimes TM \otimes TM \ldots$ where sections locally look like $A^{i_1 i_2 \ldots i_k}\frac{\partial}{\partial x^{i_1}}\otimes\frac{\partial}{\partial x^{i_2}}\ldots$ where the $A^{i_1\cdots}$ are smooth. They change upon change of coordinates as $\tilde{A}^{i_1 i_2\cdots} = A^{j_1 j_2\cdots}\frac{\partial \tilde{x}^{i_1}}{\partial x^{j_1}}\frac{\partial \tilde{x}^{i_2}}{\partial x^{j_2}}\ldots$.

Now consider $T^*M \otimes T^*M$. As a set it is $\cup_p T^*_p M \otimes T^*_p M$. Suppose (x, U) is coordinate system, then $dx^i \otimes dx^j$ span $T_q M$ for all $q \in U$. Just as before, one can use these to give a vector bundle structure to $T^*M \otimes T^*M$. Elements of this sort are called covariant tensors. A covariant tensor field is a smooth section of this bundle. Locally, a section looks like $A_{ij}(x)dx^i \otimes dx^j$ where the smooth functions $A_{ij}(x)$ transform as $\tilde{A}_{ij} = A_{kl}\frac{\partial x^k}{\partial \tilde{x}^i}\frac{\partial x^l}{\partial \tilde{x}^j}$. Likewise, one can extend this story to $T^*M \otimes T^*M \otimes T^*M \ldots$. ("Higher rank covariant tensors".)

Let us recall some linear algebra before going to vector bundles.

(1) Recall that by definition, every multilinear map from $V \times W \to \mathbb{R}$ factors as a linear map from $V \otimes W \to \mathbb{R}$.
(2) $V \otimes W \simeq W \otimes V$: Define $T(v \otimes w) = w \otimes v$ and extend it linearly to all of $V \otimes W$. It is easy to prove that this is an isomorphism.
(3) $(V \otimes W)^* \simeq V^* \otimes W^*$: Recall that given an element $T \otimes S \in V^* \otimes W^*$, one can form a linear map from $V \otimes W \to \mathbb{R}$ as $(T \otimes S)(v \otimes w) = T(v)S(w)$ (and linearly extending it from decomposable vectors to all of $V \otimes W$). By linearity, one can thus get a map $V^* \otimes W^* \to (V \otimes W)^*$. In fact, this canonical map is an isomorphism (dimension count for instance). Indeed, by induction this holds for any number of factors.
(4) Given any rank-k covariant tensor $T \in V^* \otimes V^* \ldots V^*$ and a rank l covariant tensor $S \in V^* \otimes V^* \ldots V^*$, we can get a rank $k+l$ covariant tensor $T \otimes S$ which acts on $(v_1 \otimes \ldots \otimes v^{k+l})$ as $T \otimes S(v_1 \otimes \ldots \otimes v_{k+l}) = T(v_1 \otimes v_k)S(v_{k+1} \otimes \ldots v_{k+l})$. Of course $T \otimes S \neq S \otimes T$ but it is easy to see that $T \otimes (S \otimes U) = (T \otimes S) \otimes U$.
(5) Because $V^{**} = V$ for any finite dimensional vector space, one can interpret elements of $V \otimes V \otimes V \ldots$ as linear functionals on $V^* \otimes V^* \ldots$, i.e., $(v_1 \otimes v_2 \ldots)(T_1 \otimes T_2 \ldots) = T_1(v_1)T_2(v_2)\ldots$ extended linearly.
(6) $Map(V, W) \simeq W \otimes V^*$. Indeed, given an element $w \otimes T \in W \otimes V^*$, we can define a linear map $S : V \to W$ as $S(v) = T(v)w$. By linearly extending, one has a map $W \otimes V^* \to Map(V, W)$. It is easy to prove that this is an isomorphism. Indeed, given any linear map $S : V \to W$, here is a multilinear functional on $W^* \times V$: $\tilde{S}(T, v) = T(S(v))$. Thus every linear map $V \to W$ gives a linear functional on $W^* \otimes V$, i.e., it corresponds to an element of $W \otimes V^*$.
(7) There is a canonical map - $Cont : V \otimes V^* \to \mathbb{R}$: Indeed, linearly extend the map $Cont(v \otimes T) = T(v)$. Likewise, suppose we take $V^k_l = V \otimes V \otimes$ $(k\ times)$ $\otimes V^* \ldots$ $(l\ times)$ then we have several such "contraction" maps like $Cont^3_2 : V^k_l \to V^{k-1}_{l-1}$ defined by $Cont^3_2(v_1 \otimes v_2 \ldots v_k \otimes w_1 \otimes w_2 \ldots) = w_2(v_3)v_1 \otimes v_2 \otimes v_4 \ldots \otimes w_1 \otimes w_3 \otimes \ldots$.