NOTES FOR 17 NOV (FRIDAY)

1. Recap

(1) Proved a formula for the degree in terms of preimages of a regular value.
(2) Computed the degree of the antipodal map.
(3) Proved that homotopic maps induce the same map on cohomology and hence the degree is a homotopy invariant.
(4) Proved that if a manifold deformation retracts smoothly to another, then they have the same cohomology.
(5) Proved the Hairy Ball theorem.

2. Degree of proper maps

Here is another fun application of the theory of degree.

Theorem 2.1. *Every $f(z) = z^n + a_1 z^{n-1} + \ldots$ is a polynomial from $\mathbb{C}$ to itself, then f has a complex root.*

Proof. We will not prove this fully. Instead, we sketch the main steps :
Using the stereographic projection, extend f as a smooth map from the sphere to itself. Note that $f(\infty) = \infty$. By means of a calculation, you can show that f is orientation-preserving away from $f' = 0$, i.e., away from critical points. Therefore, the degree of f is n (why?). Hence, $f^{-1}(0)$ is not empty. □

Now we return to calculating some De Rham groups.

Theorem 2.2. *For $0 < k < n-1$, we have $H^k(\mathbb{R}^n - 0) = H^k(S^{n-1}) = 0$.*

Proof. We induct on n. The base case is $\mathbb{R}^3$. Suppose ω is a closed 1-form on $\mathbb{R}^3 - (0,0,0)$. Let A be the set outside the non-negative z-axis and B be outside the non-positive z-axis. Note that A and B are star-shaped and hence $\omega = df_a$ and $\omega = df_b$ on them. Note that $d(f_a - f_b) = 0$ on $A \cap B$. Thus, $f_a = f_b + c$ where c is a constant on $A \cap B$. Now define $f = f_a$ on A and $f_b + c$ on B. Thus $\omega = df$.

If ω is a closed 1-form on $\mathbb{R}^4 - 0$, then an argument similar to the above shows that $\omega = df$. Thus, assume ω is a closed 2-form on $\mathbb{R}^4 - 0$. Then, just as above, $\omega = d\eta_a$, $\omega = d\eta_b$ with $d(\eta_a - \eta_b) = 0$. Now $H^1(A \cap B) = H^1((\mathbb{R}^3 - 0) \times \mathbb{R}) = 0$. Thus, $d(\eta_a - \eta_b) = d\lambda$ for some form λ on $A \cap B$. Thus, define $\eta = \eta_a - d(\rho_b \lambda)$ on A and $\eta = \eta_b + d(\rho_a \lambda)$ on B where ρ_a, ρ_b is a partition-of-unity. Now $\omega = d\eta$.
The general inductive step is similar. □

We do just one more calculation of cohomology.

Theorem 2.3. *For $0 \leq k < n$, we have $H^k_c(\mathbb{R}^n) = 0$.*

Proof. The $k = 0$ case is straightforward.
Let ω be a k-form with compact support. Of course, $\omega = d\eta$ for some η. We want to write $\eta = d\gamma$ outside a large ball. Suppose B is a closed ball containing the support of ω. Then the exterior of

the ball is diffeomorphic to $\mathbb{R}^n - 0$. Thus $\eta = d\gamma$ on it. Thus $\eta - d(h\gamma)$ where h is a bump function is compactly supported and $\omega = d(\eta - d(h\gamma))$. $\square$