NOTES FOR 18 AUG (FRIDAY)

1. Recap

(1) Discussed the constant rank theorem and its proof using the IFT.
(2) Applied it to maps between manifolds to conclude that there are coordinates on the base and the target so that certain maps are "standard" in these well-chosen coordinates.
(3) Gave two definitions of partitions of unity - Essentially, a partition of unity is a bunch of smooth functions adding up to 1. Either we can be given a locally finite cover, and be asked to come up with one function for every open set (whose support lies in that open set), or we can be given an arbitrary cover and be asked to come with functions of compact support, such that each function's support is in *some* open set from the cover.

2. Partition-of-unity, Whitney embedding

Firstly, we prove that producing a partition-of-unity as per the second definition. To do so, we need to prove the following topological properties of connected manifolds.

Lemma 2.1. *Suppose M is a* connected *topological manifold (Hausdorff, paracompact, locally Euclidean). Then it is second countable.*

Proof. Firstly, if M can be written as a union of countably many compact sets, then it is second countable (simply cover M) by all open coordinate balls of rational radius (call this collection of balls $\mathcal{B}$). You need only countably many of them (because each compact set needs only finitely many). These balls form a basis.
Now each element of $\mathcal{B}$ has compact closure. Choose a locally finite refinement $\mathcal{V}$ of $\mathcal{B}$. Fix $V_1 \in \mathcal{V}$. For every $V \in \mathcal{V}$, by connectedness, there is an integer N such that there is a string of open sets $V_1, V_2, \ldots, V_N = V$ such that $V_i \cap V_{i+1} \neq \phi$. By sending V to the minimum such integer we get a map $p : V \to \mathbb{Z}_+$. We shall show that the preimage of every point is finite thus showing that M is a countably union of open sets with compact closure thus proving the lemma.
Indeed, we shall do this by induction. Suppose the preimage $P_n = p^{-1}(1, \ldots, n)$ is finite by induction hypothesis. Thus $K_n = \overline{\cup_{V \in P_n} V}$ is compact. By local finiteness, around every $x \in K_n$, there is a neighbourhood U_x intersection only finitely many V. By compactness, this means that only finitely many V intersect K_n. Thus P_{n+1} is also finite completing the induction step. □

Here is a useful little lemma that we will use later (copied from Loring Tu's book).

Lemma 2.2. *Every topological connected manifold M has a compact exhaustion, i.e., a countable collection of sets $V_1 \subset \bar{V}_1 \subset V_2 \subset \bar{V}_2 \ldots$ such that $\cup V_i = \cup \bar{V}_i = M$. Also $\bar{V}_i$ are compact.*

Proof. By the proof of the previous lemma, there is a countable basis of coordinate open balls B_i of rational radius whose closures $\bar{B}_i$ are compact. Let $V_1 = B_1$. Define $i_1 \geq 2$ to be the smallest integer such that $\bar{V}_1 \subset B_1 \cup B_2 \ldots B_{i_1}$. Inductively assume that $V_1, \ldots, V_m$ have been defined. If $i_m \geq m+1, i_{m-1}$ is the smallest integer such that $\bar{V}_m \subset B_1 \cup \ldots B_{i_m}$ then set $V_{m+1} = B_1 \cup B_2 \ldots B_{i_m}$. □

Now we prove that a partition-of-unity as per the second definition exists for connected smooth manifolds :

Proof. Let V_i be the open cover as in lemma 2.2 with V_0 being the empty set. We find smooth bump functions $\psi_{i,j}$ on M with compact support on $V_{i+2} - \bar{V}_{i-1} \cap U_\alpha$ for some α such that $\sum_j \psi_{i,j} > 0$ on $\bar{V}_{i+1} - V_i$. The collection $supp(\psi_{i,j})$ will be locally finite. Since $\bar{V}_{i+1} - V_i$ cover M, $\psi = \sum_{i,j} \psi_{i,j} > 0$ on M. The functions $\phi_{i,j} = \frac{\psi_{i,j}}{\psi}$ will satisfy the desired requirements.

Indeed, for each $p \in$ the compact set $\bar{V}_{i+1} - V_i$ choose an open set U_α containing p. Let ψ be a smooth bump function > 0 on a neighbourhood W_p of p and whose support is contained in $U_\alpha \cap (\bar{V}_{i+1} - V_i)$. Only finitely many such W_{p_j} are necessary to cover the compact set $\bar{V}_{i+1} - V_i$. Take the corresponding bump functions and relabel them as $\psi_{i,j}$. These do the job. (Why ?) □

Using this partition-of-unity, we can produce one (without compact supports) satisfying the first definition :

Proof. For each k, let $\tau(k)$ be an index such that $supp(\phi_k) \subset U_{\tau(k)}$. Define $\rho_\alpha = \sum_{\tau(k)=\alpha} \phi_k$ if there is a k such that $\tau(k) = \alpha$. Otherwise set $\phi_\alpha = 0$. This will do the job. (Why ?) □

Now we shall state and prove a "baby" version of the Whitney embedding theorem.

Theorem 2.3. *Every compact smooth connected manifold M has an embedding into $\mathbb{R}^N$ for some N.*

Proof. Since M is compact, it has an open cover by finitely many coordinate open sets $U_1, \ldots, U_k$ each diffeomorphic to coordinate unit balls B_1. In fact, we may assume that k is large enough that the open sets U'_i diffeomorphic to coordinate balls $B_{1/2}$ also cover M. (Note that $\bar{U}'_i \subset U_i$.) Let $\rho_i : M \to [0, 1]$ be smooth bump functions which are 1 on $\bar{U}'_i$ and have support in U_i. Define the map $F : M \to \mathbb{R}^{kn+k}$ given by

$$F(p) = (\Phi_{U_1}(p)\rho_1(p), \Phi_{U_2}(p)\rho_2(p), \ldots, \Phi_{U_k}(p)\rho_k(p), \rho_1(p), \rho_2(p), \ldots).$$

I claim that this gives the desired embedding. Indeed,

(1) *1-1 homeomorphism* : If we prove that this smooth map is injective, then it is a homeomorphism to its image (because M is compact). If $F(p) = F(q)$, then taking an i such that $\rho_i(p) = \rho_i(q) > 0$, we see that $\Phi_{U_i}(p) = \Phi_{U_i}(q)$ which is impossible because Φ_i is a homeomorphism.
(2) *Immersion* : This is easy to see. (Why ?)

□