NOTES FOR 1 NOV (WEDNESDAY)

1. Recap

(1) Proved that orientability is equivalent to finding a nowhere vanishing top form.
(2) Defined the exterior derivative d locally and proved (using two different ways) that it is defined globally and satisfies some properties (including $d^2 = 0$).

2. Closed and exact forms, and why you should care

d commutes with pullbacks.

Proposition 2.1. *If $f : M \to N$ is smooth, and ω is a k-form on N, then $f^*(d\omega) = d(f^*\omega)$.*

Proof. Suppose $\omega = \omega_I dx^I$. Then

$$f^*(d\omega_I \wedge dx^I) = f^*(d\omega_I) \wedge \frac{\partial f^{i_1}}{\partial x^{j_1}} \frac{\partial f^{i_2}}{\partial x^{j_2}} \ldots dx^J$$
$$= f^* d\omega_I \wedge df^{i_1} \wedge df^{i_2} \ldots = d(\omega_I \circ f) \wedge df^I = d(\omega_I \circ f df^I) = df^*\omega,$$

because $d(\eta \wedge d\alpha) = d\eta \wedge d\alpha + (-1)^k \eta \wedge d^2\alpha = d\eta \wedge d\alpha$ and $f^* dg(X_p) = dg(f_* X_p) = f_* X_p(g) = X_p(g \circ f) = df^* g(X_p)$. $\square$

We introduce some terminology here : A form ω is closed if $d\omega = 0$ and exact if $\omega = d\eta$. Clearly exact forms are closed. A more interesting question is "are closed forms exact?"

In terms of vector fields in $\mathbb{R}^3$, first of all, identifying a vector field $\vec{F} = (F_1, F_2, F_3)$ with a 1-form $A = F_1 dx^1 + F_2 dx^2 + F_3 dx^3$ we see that $dA = (-\frac{\partial F_1}{\partial x^2} + \frac{\partial F_2}{\partial x^1}) dx^1 \wedge dx^2 + \ldots$. Now identifying $dx^\wedge dx^2$ with $\hat{k}$ and likewise, we see that dA is identified with $\nabla \times \vec{F}$. So if $\nabla \times \vec{F} = \vec{0}$, is $\vec{F} = \nabla f$? That is, if $dA = 0$, then is $A = df$? Likewise, if Θ is a 2-form, $\Theta = \Theta_1 dx^2 \wedge dx^3 + \ldots$ (which is identified with a vector field $\vec{T} = (T_1, T_2, T_3)$, then $d\Theta = \sum_i \frac{\partial \Theta_i}{\partial x^i} dx^1 \wedge dx^2 \wedge dx^3$ (which is identified with $\nabla . \vec{T}$. So if $d\Theta = 0$, is $\Theta = dA$? i.e. if $\nabla . \vec{T} = 0$, is $\vec{T} = \nabla \times \vec{F}$?

Actually, this question is interesting even in 2-D. If $A = F dx + G dy$, and if $dA = (\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}) dx \wedge dy = 0$, is $A = df$, i.e., $(F, G) = \nabla f$? Surely, if A is defined on all of $\mathbb{R}^2$, then define $f(x, y) = \int_0^1 (F(tx, ty)x + G(tx, ty)y) dt$. It is easy to verify that $\nabla f = (F, G)$. In fact, this is true even if replace (tx, ty) with any other path. (The function f is called the potential energy of the conservative force (F, G).)]

However, motivated by physical considerations, one can define fields on $\mathbb{R}^2 - \vec{0}$ that are not exact despite being closed. Take $A = \frac{-y}{\sqrt{x^2+y^2}} dx + \frac{x}{\sqrt{x^2+y^2}} dy$. This form is closed. However it cannot be exact. Indeed, if $A = df$, i.e., $\vec{F} = \nabla f$, then $\displaystyle\int_{S^1} \vec{F}.\vec{dr} = 2\pi \neq 0 = \int \nabla f . \vec{dr}$. Moreover, if A is any closed form, then, suppose c is a number such that $\int_{S^1} \vec{F}.\vec{dr} = c \int_{S^1} \frac{-y}{\sqrt{x^2+y^2}} dx + \frac{x}{\sqrt{x^2+y^2}} dy$, then we claim that $\omega = A - c(\frac{-y}{\sqrt{x^2+y^2}} dx + \frac{x}{\sqrt{x^2+y^2}} dy) = df$.

Sktech of proof : Indeed, suppose $\vec{W}$ is the corresponding vector field. Define $f(x, y) = \int_{\gamma(t)} \vec{W}.\vec{dr}$ where $\gamma(t)$ is any smooth 1-1 path in $\mathbb{R}^2 - \vec{0}$ taking $(1, 1)$ to (x, y). If $\tilde{\gamma}(t)$ is another such arc, then

$\int_\gamma \vec{W}.\vec{dr} - \int_{\tilde{\gamma}} \vec{W}.\vec{dr}$ can be broken up into a sum of integrals over closed curves to which one may use Green's theorem. This is annoying to make rigorous, but there is a better way to do these things.....

These considerations seem to show that whether a closed form is exact or not seems to depend on the shape of the domain. In fact,

Theorem 2.2. *Any smooth closed k-form on $\mathbb{R}^n$ is exact, i.e., if $d\omega = 0$, then $\omega = d\eta$. (Poincaré's lemma)*

Proof. If $\omega = \omega_I(x)dx^I$, then define

$$\eta = \int_0^1 t^{k-1}dt \sum_I \omega_I(tx)(x^{i_1}dx^{i_2} \wedge \ldots - x^{i_2}dx^{i_1} \wedge \ldots + \ldots) \tag{2.1}$$

The claim is that $d\eta = \omega$. Let's write sketch the proof for a 2-form in $\mathbb{R}^4$ just for illustration : $\omega = \omega_{12}dx^1 \wedge dx^2 + \ldots$. Thus

$$\eta = \int_0^1 tdt \sum_{i,j} \omega_{ij}(tx)(x^i dx^j - x^j dx^i)$$

$$d\eta = \int_0^1 dt \sum_{i,j,k} (\frac{\partial \omega_{ij}(tx)}{\partial x^k} dx^k t^2 (x^i dx^j - x^j dx^i) + 2t\omega(tx))$$

(2.2)

Since $d\omega = 0$, i.e.,

$$\frac{\partial \omega_{12}}{\partial x^3} = \frac{\partial \omega_{13}}{\partial x^2} - \frac{\partial \omega_{23}}{\partial x^1} \tag{2.3}$$

and likewise. Thus,

$$\sum_{i<j,k} \frac{\partial \omega_{ij}(tx)}{\partial x^k} dx^k (x^i dx^j - x^j dx^i) + \ldots = \frac{\partial \omega_{12}(tx)}{\partial x^3} dx^3 \wedge (x^1 dx^2 - x^2 dx^1) + \frac{\partial \omega_{13}(tx)}{\partial x^2} dx^2 \wedge (x^1 dx^3 - x^3 dx^1)$$

$$+ \frac{\partial \omega_{23}(tx)}{\partial x^1} dx^1 \wedge (x^2 dx^3 - x^3 dx^2) + \ldots = 2(\frac{\partial \omega_{12}}{\partial x^3} x^3 dx^1 \wedge dx^2 + \ldots) = \frac{d(t^2\omega(tx))}{dt} \tag{2.4}$$

□

Actually, note that we simply needed a star-shaped domain containing the origin (i.e. every point can be connected by a straight line from the origin) for the above argument to work. In fact, if you jazz it up further, it can be used to show that every contractible manifold satisfies the property that all closed forms are exact. This is there in Spivak but we shall skip it (at least for now).

3. Integration of top forms over manifolds

As discussed earlier, it appears that the only things one can integrate over manifolds are top forms ω. Also, it seemed that orientability is an important requirement.

Before going into that, in $\mathbb{R}^m$, recall the change of variables formula :

$$\int_{\mathbb{R}^m} f(x^1, \ldots, x^m) dx^1 dx^2 \ldots = \int_{\mathbb{R}^m} f(x^i(\tilde{x}^j)) |\det(\frac{\partial x^i}{\partial \tilde{x}^j})| d\tilde{x}^1 d\tilde{x}^2 \ldots \tag{3.1}$$

On the other hand, in $\mathbb{R}$, $\int_a^b f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(y))g'(y)dy$. So where did the | sign disappear ? The point is that in $\mathbb{R}^m$, we conventionally always integrate from the lower limit to the higher limit, i.e., if $g^{-1}(a) = 5$ and $g^{-1}(b) = 3$, we prefer writing $-\int_3^5 f \circ g g' dy$ and since $g' \neq 0$, this means g is decreasing and hence $-g' = |g'|$. So we chose an "orientation" (i.e. smaller number below and bigger number above) to introduce or get rid of the absolute value.