NOTES FOR 1 SEPT (FRIDAY)

1. Recap

(1) Defined the tangent space as point derivations on the algebra of germs of smooth functions.
(2) Defined the tangent bundle as a set and stated (and proved) a theorem that made into a vector bundle in an essentially unique manner. Also proved that f_* corresponded to simply the derivative matrix of f, i.e., $[Df]$ when looked at locally.

2. Vector fields, Tangent bundle, Cotangent bundle, etc

Remark 2.1. Indeed, the above proof shows that vector fields (i.e. smooth sections of TM) do satisfy the properties we want. In particular, locally, $X = X^i(x)e_i(x)$ where X^i are smooth functions.

Here are some examples of tangent bundles :

(1) As we saw, the tangent bundle of $\mathbb{R}^n$ is isomorphic to a trivial bundle $\mathbb{R}^n \times \mathbb{R}^n$.
(2) TS^1 is isomorphic to $S^1 \times \mathbb{R}$. Indeed, take the open interval of angles $\theta \in (0, 2\pi)$. This gives a coordinate chart on S^1. Choosing $\tilde{\theta} \in (-\pi, \pi)$ gives another coordinate chart and these two cover S^1. On the overlap, $\tilde{\theta} + \pi = \theta$. The coordinate vector fields $X_1 = \frac{\partial}{\partial \tilde{\theta}}$ and $X_2 = \frac{\partial}{\partial \theta}$ are related by $X_2 = \frac{\partial \tilde{\theta}}{\partial \theta} X_1 = X_1$ on the overlap. Therefore, the vector field $X = X_1$ on U_1 and X_2 on U_2 is well-defined as a smooth vector field on all of S^1. Moreover, $X \neq 0$ anywhere. Therefore TS^1 is trivial. Manifolds whose tangent bundles are trivial are called "parallelizable".
(3) The argument above shows that $T(Torus)$ is also trivial. (Why?)
(4) I claim that TS^2 is not trivial. Indeed, if it were, then there should be a vector field X that is nowhere zero. (In fact, there should be two vector fields that are linearly independent everywhere.) But this contradicts the so-called Hairy-Ball theorem (which we may see later on if time permits).
(5) If $M \subset \mathbb{R}^n$ is a submanifold given by $M = f^{-1}(0)$ where 0 is a regular value of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, then TM is isomorphic to a subset $S \subset \mathbb{R}^n \times \mathbb{R}^n$ consisting of $(\vec{p}, \vec{v})$ such that $f(\vec{p}) = 0$ and $\langle \nabla f(\vec{p}), \vec{v} \rangle = 0$.
First of all S is a manifold, in fact a vector bundle in its own right (with the projection map being $\pi_1 : S \subset \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by $\pi_1(\vec{p}, \vec{v}) = \vec{p}$). Indeed, since $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^2$ given by $F(\vec{p}, \vec{v}) = (f(\vec{p}), \langle \nabla f(\vec{p}), \vec{v} \rangle)$ has $(0, 0)$ has a regular value (why?), S is a submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ (and of course π_1 is a smooth map). Note that $\pi_1^{-1}(\vec{p})$ is the collection of all vectors $\vec{v}$ orthogonal to $\nabla f(p)$ and is hence a vector space. We will prove the local triviality property (and hence make S into a vector bundle). Indeed, suppose $\frac{\partial f}{\partial x^1}(\vec{p}) \neq 0$. By the IFT $(x^2, \ldots, x^n)$ form a coordinate chart for M on an open set U near $\vec{p}$ and $x^1 = g(x^2, \ldots, x^n)$ near p. Consider the map $h : U \times \mathbb{R}^{n-1} \to S$ given by $h(x^2 \ldots, x^n, w^1, \ldots, w^{n-1}) = (g(x^2, \ldots, x^n), x^2, \ldots, x^n, \sum_{i=1}^{n-1} \frac{\partial g}{\partial x^{i+1}} w^i, w^1, w^2, \ldots, w^{n-1})$. This is a diffeomorphism that is linear on the fibres (why?) Therefore S is a vector bundle. Moreover, the map $c_{\vec{p},\vec{v}}(t) = (g(\vec{p} + (tw^2, \ldots, tw^n)), p^2 + tw^2, \ldots, p^n + tw^n)$ (where $v^2 = w^2, v^3 = w^3 \ldots$) is a

smooth curve whose image lies on M and passes through p.

Now consider the map $t : S \to TM = \coprod_{p \in M} T_pM$ given by $t(\vec{p}, \vec{v}) = [c_{\vec{p},\vec{v}}] \in T_pM$. This is an isomorphism (why?)

Now let us look at examples of vector fields in $\mathbb{R}^n$:

(1) $\vec{X}(x, y) = (1, 1)$ on $\mathbb{R}^2$. This is a "constant" vector field.
(2) $\vec{X}(x, y) = (\frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}})$. This curls around the origin. However, strangely enough, $\nabla \times \vec{X} = 0$. (So the curl measures a very subtle form of "curling around".) Even more strangely, there is no function f such that $\nabla f = \vec{X}$. (Indeed, if there was such a function, then $\int \vec{X}.\vec{dr}$ over a circle should be 0 but it isn't.) This is a phenomenon that if time permits towards the end of this course, we may study. It goes by a fancy name called "De Rham cohomology". By the way, this vector field, when restricted to the unit circle, is a tangent vector field on S^1. It is nowhere vanishing. (Indeed, it is exactly the same as $\frac{\partial}{\partial \theta}$.)