NOTES FOR 17 NOV (FRIDAY)

1. Recap

(1) Proved the fundamental theorem of algebra.
(2) Calculated the De Rham cohomologies $H^k(S^n)$ and $H^k_c(\mathbb{R}^n)$.

2. Riemannian geometry

So far, we have studied differential topology. But we have not even proven something like the Theorema Egregium (you can't draw a to-scale map of Yeshwantpur on a piece of paper). For doing things like that, you need to do geometry. So you need to measure distances and angles. At the very least, you would need to know how long tangent vectors are.

Given a real vector space V, a symmetric positive-definite bilinear form $g : V \times V \to \mathbb{R}$ defines an inner product. Inner products satisfy the extremely important Cauchy-Schwartz inequality $|g(V,W)| \leq \sqrt{g(V,V)}\sqrt{g(W,W)}$ and the triangle inequality $\sqrt{g(V+W,V+W)} \leq \sqrt{g(V,V)} + \sqrt{g(W,W)}$. If $e_1, \ldots, e_n$ is a basis of V, then $g(V,W) = g(V^i e_i, W^j e_j) = V^i W^j g(e_i, e_j) = V^i g_{ij} W^j$ for some symmetric positive-definite matrix $g_{ij} = g(e_i, e_j)$. In terms of matrices, $g(V,W) = V^T g W$. Suppose you change your basis to $\tilde{e}_i$, then the matrix change to $\tilde{g}$ such that $\tilde{V}^T \tilde{g} \tilde{W} = V^T g W$ and hence if $\tilde{V} = PV$ for an invertible matrix P, then $P^T \tilde{g} P = g$. Sylvester's theorem in linear algebra shows that you can always choose a P so that $\tilde{g} = I$. Alternatively, by Gram-Schmidt orthogonalisation, you can always find an orthonormal basis. From now onwards, assume that $e_1, \ldots e_n$ is an orthonormal basis, i.e., $g_{ij} = \delta_{ij}$. Note that given an inner product on V, it induces one on V^*. Indeed, note that the map $T : V \to V^*$ given by $v \to g(v,)$ (i.e. $T(v)_i = g_{ij} v^j$) is an isomorphism if V is finite-dimensional. So, given an inner product on V, we have one on V^* given by $\langle T(v), T(w) \rangle = g(v,w)$. If e_i is a basis and e^{i*} the dual basis, then $\tilde{g}_{ij} = \tilde{g}(e^{i*}, e^{j*})$ can be calculated as follows : $g_{ij} v^j = (0,0,..,1...)$ implies that $v = g^{-1}(0,0,..,1,0,0,0)$. Thus, $\tilde{g}_{ij} = [e_i]^T (g^{-1})^T g g^{-1} [e_j] = [e_i]^T (g^{-1}) [e_j]$. Therefore the dual inner product as matrix is g^{-1}. Typically, we write it in coordinates as g^{ij}. Thus, $g^{ij} g_{jk} = \delta^i_k$.

Note that $g : V \times V \to \mathbb{R}$ can be thought of as an element of $(V \otimes V)^* = V^* \otimes V^*$. Indeed, $g = g_{ij}(e^{i*}) \otimes (e^{j*}) = \sum_{i=1}^n e^{i*} \otimes e^{i*}$. Also, $\tilde{g} = g^{ij} e_i \otimes e_j$ (where we are identifying $V \simeq V^{**}$).

Suppose $S \subset V$ is a subspace. Then an inner product on V induces an inner product on S. Suppose $e_1, \ldots, e_s$ is a basis of S, then, Gram-Schmidt would make this an orthonormal basis of S. One can this to an orthonormal basis of V (extend it to any basis and orthonormalise).

Notice that if we are given such an inner product, we can produce an element of $\Omega^n(V)$. Indeed, define $vol = e^{1*} \wedge e^{2*} \wedge e^{3*} \ldots$. Suppose $\tilde{e}^i$ is another orthonormal basis such that it has the same orientation as e^i, i.e., $\tilde{e}^{i*} = P^i_j e^{j*}$ where the matrix P has determinant 1, then $\tilde{e}^{1*} \wedge \ldots = \det(P) vol = vol$. If $\tilde{e}^i$ is not an orthonormal basis and suppose $g^{\mu\nu} = \tilde{g}(\tilde{e}^{\mu*}, \tilde{e}^{\nu*})$ meaning that $g^{-1} = P^T P$. Hence $\pm\sqrt{|\det(g)|} = \det(P)$. Thus, $vol = \pm\sqrt{|\det(g)|}\tilde{e}^{1*} \wedge \ldots$.

Now we carry this over to general vector bundles V over M. A metric g on a vector bundle V over M is a smooth section of $V^* \otimes V^*$ such that on each fibre it is symmetric and positive-definite. In other words, suppose e_i is a trivialisation of V over U and e^{i*} the dual trivialisation of V^* over U, then $g(p) = g_{ij}(p) e^{i*} \otimes e^{j*}$ where $g_{ij} : U \subset M \to GL(r, \mathbb{R})$ is a smooth matrix-valued function to symmetric positive-definite matrices. Now we prove that every vector bundle V admits a metric.

Theorem 2.1. *Every rank-r real vector bundle V over a manifold M admits a smooth metric g.*

Proof. Cover M with trivialising locally finite open sets U_α. Let ρ_α be a partition-of-unity subordinate to this cover. Now define $g_\alpha = \sum_{i=1}^{r} e_\alpha^{i*} \otimes e_\alpha^{i*}$. Define $g = \sum_\alpha \rho_\alpha g_\alpha$. This is clearly a smooth section which is symmetric. We only need to check that it is positive-definite. Indeed, if A and B are positive-definite, $a, b > 0$ numbers, then $(aA+bB)(v,v) = aA(v,v)+bB(v,v) \geq 0$ with equality if and only if $v = 0$. (Note that if positive-definiteness was not imposed, then g may be degenerate.) $\square$

A pleasant corollary is the following :

Corollary 2.2. *If V over M is a rank r real vector bundle, then $V \simeq V^*$ as bundles. (The isomorphism is not natural though.)*

Proof. Put a metric g on V. Then define the map $v \in V_p \to g_p(v,) \in V_p^*$. This is an isomorphism at the level of the fibres. It is also smooth, because after choosing trivialisations e_i, e^{*i} we see that $v^i \to g_{ij}(p)v^j$ which is smooth because g is so. $\square$

Another corollary is

Corollary 2.3. *If L is a line bundle over M, then L is trivial if and only if it is orientable.*

One proof of this is : V is orientable if and only if there is a nowhere vanishing section of $\Omega^r V$. In the case of a line bundle this boils down to the above. Another proof is :

Proof. If it is trivial, it is orientable. Suppose it is orientable with a smoothly varying orientation μ. Then choose a metric g on V. At every point p, there is a unique vector $s(p) \in \pi^{-1}(p)$ having unit length and pointing along $\mu(p)$. This gives a smooth nowhere vanishing section. $\square$